# ELECTRICITY AND MAGNETISM
## FOR ENGINEERS

## PART I
## ELECTRIC AND MAGNETIC CIRCUITS

ELECTRICITY AND MAGNETISM
FOR ENGINEERS
BY
HAROLD PENDER

PART I—ELECTRIC AND MAGNETIC CIRCUITS.
380 pages, 6 × 9, Illustrated.

PART II—ELECTROSTATICS AND ALTERNATING CURRENTS.
221 pages, 6 × 9, Illustrated.

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## FOR

# ENGINEERS

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## PART I

## ELECTRIC AND MAGNETIC CIRCUITS

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BY

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# PREFACE

In the following pages is given, from an engineering point of view, (1) a description of the more important effects commonly described as electric and magnetic phenomena, (2) a statement of the fundamental laws in accord with which these phenomena have been found to occur, and (3) the application of these laws to some of the simpler problems which arise in connection with the generation, transmission and utilization of electric energy.

Particular emphasis is laid upon exact *quantitative* statements of the fundamental laws or principles. Both safety and economy demand that the engineer be able to answer not only "how," but also "how much." To this end, the student of engineering should be taught to analyze, not only qualitatively, but also *quantitatively*, each problem which may be presented to him.

Most of the simpler formulas used by scientists and engineers are special cases of certain general relations, and these special formulas are applicable only under certain specific conditions. One of the most common causes of confusion on the part of the beginner arises from his attempt to apply such special formulas to cases to which they are not applicable. This is due in part to the failure in many text-books to state the *limitations* of such formulas. Particular care is therefore taken in these pages to state specifically the exact conditions under which each formula is applicable.

The procedure adopted throughout the book is to pass from simple phenomena, known to practically every school-boy, to the more complex phenomena and principles with which the engineer has to deal.

For convenience the book has been divided into two parts. Part I deals with the electric and the magnetic circuits, and Part II with electrostatics and alternating currents. Each part of the book can readily be covered in four hours of classroom work per week for a term. Part I may be looked upon as an introduction to the study of direct-current machinery, and Part II as an introduction to the study of alternating-current machinery.

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At the end of each important section are given one or more problems, illustrating the principles developed in the text. The answers to these problems are also given. The student should be required to solve each problem, and when time is available additional problems, without answers, should be assigned. It is only by the solution of numerical problems that the student can understand the full significance of the relations developed in the text.

This book covers substantially the same ground as that of the author's "Principles of Electrical Engineering," McGraw-Hill Book Company, 1911. The method of treatment, however, is distinctly different, the various laws and relations are more fully discussed, and a greater number of practical applications is given.                                   HAROLD PENDER.

PHILADELPHIA, PA.,
    *Oct. 3, 1918.*

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# ELECTRICITY AND MAGNETISM FOR ENGINEERS

## PART I
### ELECTRIC AND MAGNETIC CIRCUITS

### I

### FORCE, WORK AND ENERGY

**1. Introduction.**—Electricity manifests itself in the production of forces between material bodies and in the production of various forms of energy. A knowledge of the terms employed in describing the effects of mechanical forces and energy, and of the fundamental relations between matter, force and energy, is therefore essential to an understanding of the facts and "laws" of electricity and magnetism. For the convenience of the reader a brief review of these terms and principles is here given.

**2. Scalar and Vector Quantities.**—A quantity which can be completely specified by a magnitude, such as mass, work, temperature, etc., is called a "scalar" quantity. A scalar quantity may be either positive or negative. In the first case it is represented by a positive number and in the latter case by a negative number. Two or more scalar quantities are added and subtracted according to the ordinary laws of algebra.

A quantity, such as a force, which requires for its complete representation not only a magnitude but a definite direction in space, is called a "vector" quantity. A vector quantity can always be represented by a straight line of definite length drawn in a definite direction in space. The line representing a vector quantity is called a "vector."

A vector may be conveniently represented by two letters which designate its ends. The convention universally adopted is to write as the first letter that letter which designates the point *from which* the vector is drawn. For example, by the vector $\overline{AB}$ is meant the straight line drawn from $A$ to $B$. Similarly, by the vector $\overline{BA}$ is meant the straight line drawn from $B$ to $A$.

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When a vector is represented by a line $\overline{AB}$, the direction from $A$ to $B$ is called the "positive sense" of this line.

The length of a line may also be represented by a single letter or number, and the positive sense of this line by an arrow placed on the line pointing in its positive sense. Hence if $V$ is used to designate the vector $\overline{AB}$, then the length and sense of the vector drawn from $A$ to $B$ may be represented by the algebraic expression $+V$ and the length and sense of the vector drawn from $B$ to $A$ by the algebraic expression $-V$.



FIG. 1.

The relative direction of two vectors, such as $\overline{AB}$ and $\overline{AC}$ in Fig. 2, is conveniently specified as follows: When the angle measured around from $\overline{AB}$ to $\overline{AC}$ in the *counter-clockwise direction* is less than 180 degrees, the vector $\overline{AC}$ is said to "lead" $\overline{AB}$ by this angle, and conversely, the vector $\overline{AB}$ is said to "lag" $\overline{AC}$ by this same angle. When the angle measured around from $\overline{AB}$ to $\overline{AC}$ in the counter-clockwise direction is greater than 180 degrees, then the angle measured around from $\overline{AB}$ to $\overline{AC}$ in the clockwise direction must be less than 180 degrees, and therefore $\overline{AB}$ leads $\overline{AC}$ by this latter angle, or $\overline{AC}$ lags $\overline{AB}$ by this same angle. A
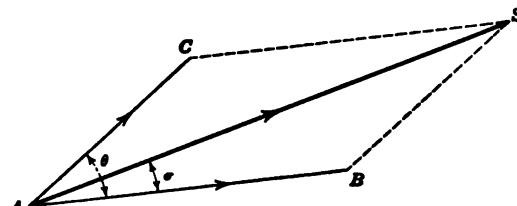


FIG. 2.

lead of $\phi$ degrees is therefore equivalent to a lag of $\theta = 360 - \phi$ degrees, and a lag of $\phi$ degrees is equivalent to a lead of $\theta = 360 - \phi$ degrees. Hence it is always possible, by employing the conceptions of lead and lag, to specify the relative direction of two vectors by an angle less than 180 degrees.

By the "vector sum" of two or more vectors, such as $\overline{AB}$ and $\overline{AC}$ in Fig. 2, is meant vector equal to the diagonal $\overline{AS}$ of the parallelogram formed by drawing through $C$ a line $\overline{CS}$ parallel to $\overline{AB}$ and through $B$ a line $\overline{BS}$ parallel to $\overline{AC}$. If the angle

by which $\overline{AC}$ leads $\overline{AB}$ is $\theta$, then the vector sum of $\overline{AB}$ and $\overline{AC}$ results in a vector of length

$$\overline{AS} = \sqrt{\overline{AB}^2 + \overline{AC}^2 + 2\,\overline{AB}.\overline{AC}\,\cos\theta} \qquad (1)$$

*leading* $\overline{AB}$ by the angle

$$\sigma = \tan^{-1}\left[\frac{\overline{AC}\,\sin\theta}{\overline{AB} + \overline{AC}\,\cos\theta}\right] \qquad (1a)$$

From these relations it follows that the vector sum of two vectors which are equal in length and opposite in sense, such as $\overline{AB}$ and $\overline{BA}$ (Fig. 1) is zero, for the difference in direction between these two vectors is 180 degrees, and therefore $\cos\theta = -1$. This fact may be expressed mathematically by the relation

$$0 = \overline{AB} + \overline{BA}$$

or

$$\overline{BA} = -\overline{AB}$$

By the "vector subtraction" of a vector $\overline{AC}$ from a vector $\overline{AB}$ is meant the vector addition of $-\overline{AC}$ to $\overline{AB}$, which, from the
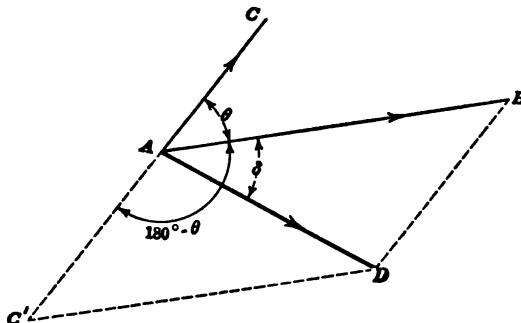


FIG. 3.

relation just deduced, is equivalent to turning the vector $\overline{AC}$ through 180 degrees and adding it to $\overline{AB}$, as shown in Fig. 3. Hence the vector subtraction of $\overline{AC}$ from any vector $\overline{AB}$ results in a vector of length

$$\overline{AD} = \sqrt{\overline{AB}^2 + \overline{AC}^2 - 2\,AB.\overline{AC}\,\cos\theta} \qquad (2)$$

*lagging* $\overline{AB}$ by the angle

$$\delta = \tan^{-1}\left[\frac{\overline{AC}\,\sin\theta}{\overline{AB} - \overline{AC}\,\cos\theta}\right] \qquad (2a)$$

It is also of interest to note that the length and direction of this vector difference are respectively the same as the length and direction of diagonal $\overline{CB}$ of the parallelogram in Fig. 2.

A special case of vector subtraction of particular importance is that of the vector difference of two vector quantities which have equal magnitudes and which differ in direction by an infinitely small angle. Let $V$ be the magnitude of each of the two vectors $\overline{AB}$ and $\overline{AC}$ in Fig. 4, let $d\theta$ be the angle between them,
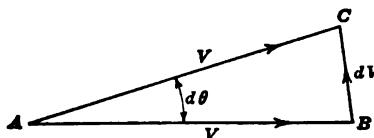


FIG. 4.

expressed in radians, and let $dV$ represent the vector difference $\overline{BC}$. Since the angle between $\overline{AB}$ and $\overline{AC}$ is infinitely small, the ratio of the length of the line $\overline{BC}$ to the length of the line $\overline{AB}$ is equal to the angle $d\theta$ in radians.[1] Hence

$$dV = V d\theta \qquad (2b)$$

Also, since $d\theta$ is infinitely small, the line $\overline{BC}$ is perpendicular to $AB$. Hence, from Fig. 4, the vector difference $dV$ is perpendicular to the vector $V$ and is in the direction of the positive sense of the angle $d\theta$.
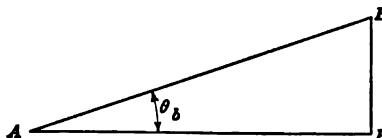


FIG. 5.

It may be readily shown that any vector $\overline{AB}$ may be considered as the sum or "resultant" of two or more component vectors, provided these component vectors have such lengths and directions that when placed end to end they form a continuous path from $A$ to $B$, such that a point in passing from $A$ to $B$ over this path moves along each component in its positive sense. In particular, it is frequently convenient to consider a vector $\overline{AB}$

[1] The measure of an angle in radians is by definition the ratio of the arc which subtends this angle to the radius of this arc. When the angle is infinitely small the chord of this arc is equal to the arc itself.

as the sum of two mutually perpendicular components, such as $\overline{AP}$ and $\overline{PB}$ in Fig. 5. If $\theta_b$ is the angle by which $\overline{AB}$ leads the component $\overline{AP}$, then the two components $\overline{AP}$ and $\overline{PB}$ are

$$\overline{AP} = \overline{AB} \cos \theta_b \qquad (3)$$
$$\overline{PB} = \overline{AB} \sin \theta_b \qquad (3a)$$

Conversely, when the two mutually perpendicular components $\overline{AP}$ and $\overline{PB}$ are known, then the resultant $AB$ has the length

$$\overline{AB} = \sqrt{\overline{AP}^2 + \overline{PB}^2} \qquad (4)$$

and leads the component $\overline{AC}$ by the angle

$$\theta_b = \tan^{-1}\left(\frac{\overline{PB}}{\overline{AP}}\right) \qquad (4a)$$

These relations lead to another way of expressing the sum of two vectors. Let $\overline{AB}$ and $\overline{AC}$ be the two vectors. At the end of $\overline{AB}$ lay off $\overline{BS}$ equal in length to $\overline{AC}$ and in the same direction
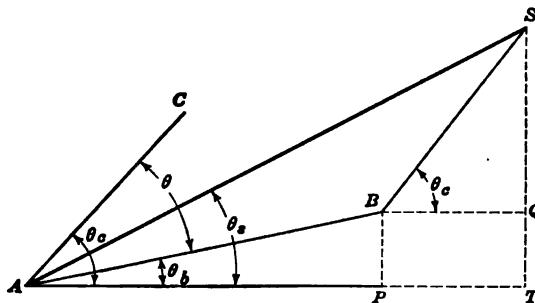


Fig. 6.

as $\overline{AC}$, and resolve $\overline{AB}$ and $\overline{BS}$ into their horizontal and vertical components. Then, using the notation indicated in Fig. 6, the vector sum of $\overline{AB}$ and $\overline{BS}$ is a vector of length

$$\overline{AS} = \sqrt{\overline{AT}^2 + \overline{TS}^2} \qquad (5)$$

which leads $\overline{AP}$ by the angle $\theta_s$, where

$$\tan \theta_s = \frac{\overline{TS}}{\overline{AT}} \qquad (5a)$$

The values of $\overline{TS}$ and $\overline{AT}$ are

$$\overline{TS} = \overline{PB} + \overline{QS} = \overline{AB} \sin \theta_b + \overline{AC} \sin \theta_c$$
$$\overline{AT} = \overline{AP} + \overline{BQ} = \overline{AB} \cos \theta_b + \overline{AC} \cos \theta_c$$

That is, the component, in any direction, of the resultant of two or more vectors is the sum of the components, in this same direction, of the individual vectors.

By calculating first the angle $\theta_s$ from equation (5a), the evaluation of the radical expression may be avoided by calculating the vector sum $\overline{AS}$ from the formula

$$\overline{AS} = \frac{\overline{AT}}{\cos \theta_s} \tag{5b}$$

A very convenient table of trigonometric functions for use in such calculations is given in PENDER's HANDBOOK FOR ELECTRICAL ENGINEERS, pages 1709 to 1715.

Problem 1.—Two vectors $A$ and $B$, 5 and 10 inches in length respectively, make an angle of 30 degrees with each other, the shorter vector $A$ leading the longer one.

(a) What is the vector sum of these two vectors?  (b) By what angle does this vector sum, or resultant, lead the longer vector?  (c) What is the vector difference of $A$ minus $B$?  (d) By what angle does this vector difference lead $A$?  (e) By what angle does the vector difference $B$ minus $A$ lag $A$?  (f) What is the component of $A$ in the direction of $B$?  (g) What is the component of $B$ perpendicular to $A$, and does it lead or lag $A$?  (h) If the angle by which $A$ leads $B$ is changing at the rate of 157 radians per second, what is the length and direction, with respect to $A$, of the vector which is equal to the rate of change of the vector $A$ with respect to time?  Draw a vector diagram to illustrate the answer to each of the above questions.

Answer.—(a) 14.55 inches.  (b) 9.9 degrees.  (c) 6.20 inches.  (d) 156.2 degrees.  (e) 23.8 degrees.  (f) 4.33 inches.  (g) 5 inches, lagging.  (h) 785 inches, leading $A$ by 90 degrees.

Problem 2.—A vector $A$ which has a length of 12 inches leads a reference line $\overline{OX}$ by 40 degrees.  Another vector $B$ has a component in the direction of $\overline{OX}$ equal to 0.6 inch, and a component 90 degrees ahead of $\overline{OX}$ equal to 1.5 inches.

(a) What is the component in the direction of $\overline{OX}$ of the vector sum of $A$ and $B$?  (b) What is the component of this resultant 90 degrees ahead of $\overline{OX}$?  (c) What is the angle by which this resultant leads $\overline{OX}$?  (d) What is the magnitude of this resultant?

Answer.—(a) 9.79 inches.  (b) 9.21 inches.  (c) 43.3 degrees.  (d) 13.45 inches.

**3. Linear Displacement, Velocity and Acceleration.**—When a particle moves from a point $A$ to some other point $B$, fixed relative to the point $A$, the straight line drawn from $A$ to $B$ is called the "linear displacement" of this particle relative to $A$.

The rate of change with respect to time in the linear displacement of a particle relative to any point $A$ is called the "linear

velocity" of this particle with respect to the point $A$. When this rate of change is uniform, the velocity is

$$v = \frac{l}{t} \qquad (6)$$

where $l$ is the displacement in time $t$. When the rate of change in the displacement of the particle is not uniform, then the velocity at any instant $t$ is

$$v = \frac{dl}{dt} \qquad (6a)$$

where $dl$ represents the change in the displacement in an infinitesimal interval of time $dt$ measured from this instant.

The rate of change with respect to time in the linear velocity of a particle, relative to any point $A$, is called the "linear acceleration" of this particle with respect to the point $A$. When the rate of change of velocity is uniform, then the acceleration is

$$a = \frac{v_1 - v_2}{t} \qquad (7)$$

where $v_1 - v_2$ is the increase in velocity in time $t$. When this rate of change is not uniform the acceleration is

$$a = \frac{dv}{dt} = \frac{d^2l}{dt^2} \qquad (7a)$$

Linear displacement, velocity and acceleration are all vector quantities, *i.e.*, each has a direction as well as a magnitude. The term "speed" is commonly used to designate the magnitude of a velocity.

Displacements, velocities and acceleration therefore cannot be added and subtracted algebraically, but must be added and subtracted *vectorially*. In particular, when the *direction* of the velocity of a point changes, $v_1 - v_2$ in equation (7) and $dv$ in equation (7a) must be interpreted as *vector* differences. For example, when a point is moving in a circular path of radius $r$ at a constant speed $V$, the direction of its velocity will change during any infinitesimal interval of time $dt$ by the angle $d\theta = \frac{V dt}{r}$. Hence, from equation (2b), the change in the linear velocity of this particle in time $dt$ is $dV = V d\theta = \frac{V^2 dt}{r}$. From equation (7a) the linear acceleration of this particle at this instant is therefore

$$a = \frac{V^2}{r} \qquad (7b)$$

and, since $dV$ is perpendicular to the direction of the velocity $V$ and in the positive sense of the angle $d\theta$, the direction of this acceleration is toward the center of the circle.  In this formula $V$ and $r$ must be expressed in the same units of length; for example, $V$ in miles per hour and $r$ in miles, in which case $a$ will be in miles per hour per hour.

**Problem 3.**—A locomotive is traveling at the rate of 60 miles an hour around a curve of 1000 feet radius.

(a) What is the speed of this locomotive in meters per second?  (b) In feet per second?  (c) What is the direction and amount of its acceleration in miles per hour per second?  (d) In feet per second per second?

*Answer.*—(a) 26.8 meters per second.  (b) 88.0 feet per second.  (c) 5.28 miles per hour per second, toward the center of the curve.  (d) 7.75 feet per second per second.

**4. Angular Displacement, Velocity and Displacement.**—Let $OX$ be any arbitrarily chosen axis, and imagine a plane drawn through this axis and a moving particle.  The angle through which this plane turns when the particle moves from any point $A$ to some other point $B$, fixed with respect to the point $A$ and the axis $OX$, is called the "angular displacement" of the particle about this axis relative to the point $A$.

The rate of change with respect to time in the angular displacement of a particle about a given axis $OX$ is called the angular velocity of the particle about this axis, and the rate of change in this angular velocity with respect to time is called its angular acceleration about this axis.

When the angular displacement is changing at a constant rate, the angular velocity is

$$\omega = \frac{\theta}{t} \tag{8}$$

where $\theta$ is angular displacement in time $t$.  When the rate of change in the angular displacement is not uniform, then the angular velocity is

$$\omega = \frac{d\theta}{dt} \tag{8a}$$

Similarly, when the angular velocity is changing at a constant rate, the angular acceleration is

$$\alpha = \frac{\omega_1 - \omega_2}{t} \tag{9}$$

where $\omega_1 - \omega_2$ is the increase in the angular velocity in time $t$.

When the rate of change in the angular velocity is not uniform, then the angular acceleration is

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\omega}{dt^2} \qquad (9a)$$

The fundamental unit of angular displacement is the radian, and the corresponding units of angular velocity and angular acceleration are respectively 1 radian per second and 1 radian per second per second. In the case of a rigid body, rotating about a fixed axis, the unit of angular velocity commonly employed is one complete revolution per minute. One complete revolution is equal to $2\pi$ radians, and hence one revolution per minute is equal to $\frac{2\pi}{60}$ radians per second.

**Problem 4.**—The armature (*i.e.*, the rotating member) of an electric motor, starting from rest, reaches a speed of 750 revolutions per minute (abbreviated r.p.m.) in 10 seconds, the change in velocity being at a uniform rate.

(*a*) What is the angular acceleration of the armature in radians per second per second? (*b*) How many revolutions will the armature make before reaching the speed of 750 r.p.m.?

*Answer.*—(*a*) 7.854 radians per second per second. (*b*) 62.5 revolutions.

**5. Mass.**—The quantity of matter in a body, called its "mass," can be defined only in terms of some effect produced on the body by some other body or bodies external to it. It is a fact of experiment that two bodies, which appear to our senses to be identical in every respect, will exactly counterbalance each other when suspended one from each end of an equal-armed balance in a vacuum. As an arbitrary assumption, the mass of any two bodies may then be defined as equal, irrespective of their volume, shape or chemical composition, if, when suspended in a vacuum, one from each end of an equal-armed balance, and each free from all external influences other than that due to the earth, there is no tipping of the beam of the balance from its original position of equilibrium.

This is an entirely arbitrary definition, but it has been found that mass as thus defined is a fundamental property of matter. Any arbitrary portion of matter may be taken as the unit of mass; the mass of any given portion of matter may then be expressed as the number of such equal units, which taken together, and suspended from one arm of an equal-armed balance in a

vacuum, will just counterbalance the given body suspended from the other arm. Note that mass is a scalar quantity.

**6. Systems of Units.**—The standard unit of mass to which all other units of mass are referred is a certain platinum-iridium cylinder, known as the International Kilogram, preserved at the International Bureau of Weights and Measures near Paris. The standard unit of length is similarly the length, at a temperature of 0° C., of a certain platinum-iridium bar, known as the International Meter, preserved at the same place.

As the standard unit of time is taken the mean solar day. By a solar day is meant the interval of time between two successive transits of the sun across the meridian of the earth at the point of observation. This interval varies in length at different times during the year, but the *mean* solar day, *i.e.*, the average length of this interval for each year, remains constant; or more precisely, may be assumed to remain constant.

Submultiples of the meter, kilogram and solar year are generally employed as more convenient units. The three standard submultiples are:

$$1 \text{ centimeter} = \frac{1}{100} \text{ meter.}$$

$$1 \text{ gram} = \frac{1}{1000} \text{ kilogram.}$$

$$1 \text{ second} = \frac{1}{86,400} \text{ mean solar day.}$$

In terms of these three units, namely, the centimeter, the gram and the second, a unit may be specified for every measurable quantity. These three units are therefore called the three "fundamental" units, and any unit specified in terms of these three fundamental units is called a "derived" unit.

The system of units specified in terms of the centimeter, gram and second in such a manner that the fundamental units appear in the specification only as unity or as integral multiples of 10, 100, 1000, etc., or their reciprocals, is called the "c.g.s. metric system" of units.

Multiple and submultiple units in the c.g.s. system are designated by the following prefixes:

$$\text{micro} = \frac{1}{1,000,000} = 10^{-6}$$

$$\text{milli} = \frac{1}{1000} = 10^{-3}$$

$$\text{centi} = \frac{1}{100} = 10^{-2}$$

$$\text{deci} = \frac{1}{10} = 10^{-1}$$

$$\text{deka} = 10$$

$$\text{hecto} = 100 = 10^2$$

$$\text{kilo} = 1000 = 10^3$$

$$\text{myria} = 10,000 = 10^4$$

$$\text{mega} = 1,000,000 = 10^6$$

For example, a millimeter is one-thousandth of a meter or one-tenth of a centimeter; a kilometer is 1000 meters; a decigram is one-tenth of a gram or one ten-thousandth of a kilogram, etc.

The English, or foot-pound-second, system of units was originally based upon arbitrary units of mass and length, independent of the kilogram and meter, namely, the standard pound and the standard yard. However, the present U. S. legal definitions (Act of Congress, July 28, 1866) are

$$1 \text{ yard } = \frac{3600}{3937} \text{ meter}$$

and

$$1 \text{ pound } = \frac{1}{2.2046} \text{ kilogram.}$$

(NOTE.—In 1893 the Superintendent of Weights and Measures, with the approval of the Secretary of the Treasury, declared the pound to be more precisely $\frac{1}{2.204622}$ kilogram.)

The interrelations of the various units used in engineering, and their multiples and submultiples in the various systems in use in this country, may be found in any engineers' handbook.

**7. Density and Specific Gravity.**—The "density" of a uniform substance is defined as the mass of this substance per unit volume. In the c.g.s. system, density is the mass in grams of 1 cubic centimeter of the substance. When the substance is not uniform, its density at any point is defined as the mass of an infinitesimally small volume taken about the point, divided by this volume; *i.e.*, calling $dv$ the volume and $dm$ the mass of this volume, the density is

$$\delta = \frac{dm}{dv} \tag{10}$$

The "specific gravity" of a substance is defined as the ratio of the mass or weight of a given volume of the substance to the mass or weight of an equal volume of water at standard temperature. Sixty-two degrees Fahrenheit is usually taken as the standard temperature, although there is no general agreement on this point. Density in the c.g.s. system and specific gravity are practically numerically equal.

**8. Center of Mass.**—A body of mass $M$ which has any size or shape may be considered as made up of a number of small particles of masses $m_1$, $m_2$, $m_3$, etc., such that $m_1 + m_2 + m_3 + \ldots = M$. These particles may be considered as small as desired, that is, each particle may be considered so small that it occupies but a point in space.

Consider three mutually perpendicular planes $X$, $Y$, $Z$, fixed in space, and represent by $x_1$, $y_1$, and $z_1$ the perpendicular distances of the particle $m_1$ from these planes respectively, and by $x_2$, $y_2$, and $z_2$ the perpendicular distances of the particle $m_2$ from these three planes respectively, and so on for the other particles; then the point whose distances from these three planes are respectively

$$x = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots}{M}$$

$$y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \ldots}{M} \qquad (11)$$

$$z = \frac{m_1 z_1 + m_2 z_2 + m_3 z_3 + \ldots}{M}$$

is defined as the "center of mass" of the body.

The center of mass of a body is therefore the point the distance of which from each of three mutually perpendicular planes is the average distance of the matter in the body from each of these planes. It can be shown that the position of the center of mass of a body relative to any point in the body is independent of the position of the planes of reference.

The center of mass of a system of any number of bodies is defined in exactly the same manner, except that $M$ in this case is taken as the *total* mass of *all* the bodies.

The center of mass of a body or system of bodies is also called its "center of gravity," or "center of inertia."

**9. Moment of Inertia and Radius of Gyration.**—Consider any axis of reference $X$ and any particle of mass $m$ at a distance $r$

from this axis; then the product $mr^2$ is called the moment of inertia of the particle $m$ about the axis $X$. The moment of inertia of an extended body or system of bodies about any axis is

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots \qquad (12)$$

the summation including the products $mr^2$ for all the particles of the body or system of bodies.

By the "radius of gyration" of a solid body, about any axis of rotation, is meant a distance such that the product of the total mass $M$ of the body times the square of this distance is equal to the moment of inertia $I$ of the body about this axis, *i.e.*, a distance $\rho$ such that

$$I = M\rho^2 \qquad (13)$$

**Problem 5.**—The radius of gyration of a cylinder about its own axis is equal to one-fourth of the diameter of the cylinder. The specific gravity of iron is 7.7. What is the moment of inertia, about its own axis, of an iron cylinder 5 feet in diameter and 4 feet long? Give answer in pound-foot units.

*Answer.*—59,100 pounds-times-feet-squared.

**10. Linear Momentum and Angular Momentum.**—When the center of mass of a body of mass $m$ is moving with a linear velocity $v$ with respect to any point $P$, the product $mv$ is called the linear momentum of the body with respect to this point, viz.,

$$\text{Linear momentum} = Mv \qquad (14)$$

When a body has a moment of inertia $I$ about any axis $X$ and is rotating about this axis with an angular velocity $\omega$ with respect to any point $P$, the product $I\omega$ is called the angular momentum of the body about this axis with respect to the given point, viz.,

$$\text{Angular momentum} = I\omega \qquad (14a)$$

**11. Force and Weight.**—In general terms, a force is that which produces, or tends to produce, a change in the state of rest or motion of a body, or "a force is pull or a push." The nature of force is not thoroughly understood, but the effects of a force, *e.g.*, change in motion, the extension or compression of a spring, etc., are readily measured.

It is a fundamental fact of experience that every particle of matter exerts a force on every other particle of matter, the value of this force depending in general upon the nature of the two

particles, their relative position and their state of motion. As the measure of the force produced by any particle $A$ on another particle $B$ is taken the product of the mass of $A$ by the linear acceleration, with respect to the center of mass of the system formed by these two particles, which this force, acting alone, would give to $B$. That is, calling $m$ the mass of $A$, $f$ the force exerted on $A$ by any other particle $B$, and $a$ the linear acceleration, with respect to the center of mass of the system formed by $A$ and $B$, which this force acting alone would give to $A$, then, by definition,

$$f = ma \tag{15}$$

The facts of experiment and observation expressed in terms of this definition lead to the conclusion that when any particle $B$ exerts a force on any other particle $A$, then the second particle $A$ exerts an equal and opposite force $B$. This fundamental law of nature is usually expressed by the statement that **action and reaction are equal and opposite.**

The commonest force with which one has to deal is the force exerted by the earth on all bodies at or near its surface, as shown by the tendency of every body to fall toward the earth. This force is usually referred to as the "force of gravitation." Experiment shows that when a body is allowed to fall freely (*i.e.*, when its motion is neither aided nor impeded by any other force), it will fall vertically with a velocity which increases at a uniform rate; that is, with a *constant acceleration* with respect to the surface of the earth. Moreover, experiment shows that at any given place at the earth's surface this acceleration has a constant value, irrespective of the mass, size, shape or material of the falling body. This "acceleration due to gravity" is usually represented by the symbol $g$.

From the definition of the measure of a force, equation (15), it therefore follows that the force of gravitation exerted by the earth on a particle of mass $m$ is

$$f_w = mg \tag{15a}$$

This force is called the "weight" of the particle.

Experiment shows that the value of $g$, although constant for all bodies at any one place, varies slightly with the latitude and elevation of the place in question. The "standard" value

of $g$ adopted by international agreement,[1] which may be designated $g_o$, is

$$g_o = 980.665 \text{ cm. per sec. per sec.}$$
$$g_e = 32.1739 \text{ ft. per sec. per sec.}$$

The value of $g$ for any other location varies but slightly from this value, being, at sea level, approximately 0.3 per cent. less at the equator, and 0.3 per cent. greater at the poles, and decreasing at the rate of about 0.01 per cent. per 1000 feet increase in elevation.

The direction of any force acting on a particle is defined as the direction of the acceleration which this force, if acting alone, would give to this particle. The direction of the gravitational force acting on a particle is always toward the center of the earth, *i.e.*, is always vertically downward.

Experiment shows that when a particle is acted upon by two or more forces, the resultant motion of the particle is the same as would be produced were it acted upon by a single force equal to the **vector sum** of these separate forces. In particular, when the acceleration of the particle is zero, *i.e.*, when there is no change in its velocity, then the vector sum of all the forces acting upon it is zero. Conversely, when the vector sum of the forces acting on a particle is zero, the velocity of the particle is constant.

In the case of an extended body (as contrasted with a single particle), the external force applied to it may be exerted directly only at one or more points. For example, the force exerted on a pulley by the belt which drives it is exerted directly only upon the points on the periphery of the pulley which are in contact at any instant with the belt. The point at which any external force is applied to a body is called the "point of application" of this force.

It may be shown, as a consequence of the definitions and relations stated in the preceding paragraphs, that the linear acceleration of the center of mass of a rigid body (and in fact of any system of particles) is always equal to the **vector sum** of all the external forces acting upon this body (irrespective of their points of application), divided by the total mass of this body. That is, calling $M$ the total mass of the body, $\alpha$ the linear acceleration of

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[1] *Troisième Conf. Gen. des Poids et Mes.*, 1901, p. 66.

its center of mass, then the vector sum of all the external forces acting upon this body is

$$F = Ma \tag{15b}$$

**12. Units of Force.**—The rational unit of force is that force which will give unit linear acceleration to unit mass. This unit of force is called the "absolute" unit of force. When the mass is expressed in grams and the acceleration in centimeters per second per second, the corresponding absolute unit of force is called the "dyne;" when the mass is expressed in pounds and the acceleration in feet per second per second, the corresponding absolute unit of force is called the "poundal."

The relation expressed by equation (15a), however, suggests another unit of force which for many purposes is very convenient, and in fact is the unit ordinarily used by engineers. This unit, called the "gravitational" unit of force, is the force exerted by the earth on unit mass. Since the value of the force varies with latitude and elevation, it is also necessary to specify a definite place of measurement of this unit force, or better, a specific value of the acceleration $g$, which shall be used in evaluating this force in terms of the absolute unit. The value of $g$ adopted is the "standard" value $g_o$ specified in the preceding article.

The unit of force in the gravitational system is given the same name as the corresponding unit of mass. For example, by a force of 10 pounds is meant a force equal to that which would be exerted by the earth on a mass of 10 pounds at a place where the acceleration due to gravity is 32.1739 feet per second per second. Hence, the resultant force $F$ acting on a body of total mass $M$ when its center of mass has an acceleration $a$ is

$$F = M a \qquad \text{absolute units} \tag{15c}$$

or

$$F = \frac{M}{g_o} a \qquad \text{gravitational units} \tag{15d}$$

where $g_o$ is the *numerical* value of the acceleration due to gravity, in the system of units employed.

Problem 6.—Referring to Problem 3, if the locomotive weighs 100 tons (200,000 pounds):

(a) What would be the centrifugal force exerted by the flanges of its wheel against the outer rail on the curve, assuming that both rails are in the same horizontal plane and that the outer wheels do not rise? (b) If this force is exerted against 15 spikes holding the rail to the ties, what would be the

force tending to shear off each spike? (c) How could this shearing force be reduced or eliminated? (d) If the outer rail is elevated 6 inches with respect to the inner rail, what would be the shearing force on each spike? The distance between rails is 4 feet, 8.5 inches.

*Answer.*—(a) 48,200 pounds. (b) 3210 pounds. (c) By elevating the outer rail. (d) 1794 pounds.

**Problem 7.**—The per cent. grade of a railroad is the number of feet vertical rise in a horizontal distance of 100 feet.

(a) What is the total force, or "tractive effort," required to draw a train weighing 800 tons at constant speed up a grade of 3 per cent., assuming that the frictional resistance of the train (including all rolling friction and air friction) is 10 pounds per ton of total train weight? (b) What force is required to give a ton a linear acceleration of 1 mile per hour per second? (c) If the maximum tractive effort developed by the locomotive of this train is 60,000 pounds, at what rate can it accelerate the train up the given grade of 3 per cent.? (Neglect the inertia of the rotating wheels.)

*Answer.*—(a) 56,000 pounds. (b) 91.2 pounds per ton. (c) 0.0548 miles per hour per second.

**13. Pressure.**—The perpendicular component of the force *per unit area* exerted on any surface is called the pressure on that surface. Let $df$ be the perpendicular or **normal component of the force** acting on an area $dA$, then the pressure at $dS$ is

$$p = \frac{df}{dS} \tag{16}$$

The term pressure is sometimes used as a synonym for force, and what is here defined as pressure is then called the "intensity of pressure."

The rational unit of pressure is force per unit area, *e.g.*, dynes per square centimeter, or pounds per square inch. However, a number of arbitrary units of pressure are employed, such as the pressure at the base of a column of mercury 1 centimeter in height, due to the weight of the particles which constitute this column.

The atmosphere exerts a pressure of approximately 14.7 pounds per square inch upon every surface with which it is in contact (the exact value depending upon weather conditions and the elevation of the point of observation). The total pressure on a surface, including that applied by any artificial means and that of the atmosphere, is called the "absolute pressure" on the surface

**14. Torque or Moment of a Force.**—Consider any axis and any force acting on a particle at a perpendicular distance $r$ from

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this axis; let $f$ be the component of this force perpendicular to the plane through this axis and the particle. Then the product

$$T = rf \qquad (17)$$

is called the "moment of the force," or the "torque," due to this force about the given axis. The distance $r$ is called the "lever arm," or simply the "arm," of the force about this axis.

When the force is expressed in absolute units the torque is likewise said to be in absolute units, and when the force is expressed in gravitational units the torque is in gravitational units. The units of torque are of the same nature, or dimensions, as the units of energy (see Article 15), but the two words in the compound names of the energy units are usually reversed. For example, the foot-pound is an energy unit, but the corresponding unit of torque is called the pound-foot.

A given torque is frequently referred to as a torque of so many units at unit radius; *e.g.*, a torque of 10 pounds at 1 foot radius. However, torque is not measured in pounds but in pound-feet. Hence, a preferable expression would be a torque *equivalent to* a force of 10 pounds at 1 foot radius.

From the definitions and relations stated in the preceding articles it may be shown that the angular acceleration of a *rigid* body about any axis *through its center of mass*, in radians per second per second, is always equal to the algebraic sum of the torques about this axis due to the external forces acting upon the particles of which the body consists, divided by the moment of inertia of this body about this axis. That is, calling $I$ the moment of inertia of the body about any given axis through its center of mass, $\alpha$ the angular acceleration of any particle in it about this axis, in radians per second per second, then the algebraic sum of all the torques due to the external forces acting upon this body is

$$T = I \alpha \qquad \text{absolute units} \qquad (18)$$

or

$$T = \frac{I}{g_o} \alpha \qquad \text{gravitational units} \qquad (18a)$$

**Problem 8.**—The tangential force exerted by a brake on a pulley which is rotating at a speed of 500 r.p.m. is 200 pounds. The diameter of the pulley is 1 foot. What is the torque exerted on this pulley?

*Answer.*—100 pound-feet.

**15. Work and Energy.**—When a particle moves under the action of a force, "work" is said to be done on it by this force, or by the agent which produces this force. Since distance is always involved in motion, work involves both distance and force. Moreover, since a force has no component at right angles to its direction, it has no tendency to either aid or oppose a motion at right angles to its direction. Hence, work involves also the relative direction of the force with respect to the motion of its point of application.

As a measure of the work done by a force when the particle or point at which it acts is displaced, is taken the product of this displacement by the **component of this force** in the direction of this displacement. That is, the work done by a force $f$ when the particle on which it acts moves a distance $dl$ is

$$dW = (f \cos \theta) \, dl \qquad\qquad (19)$$
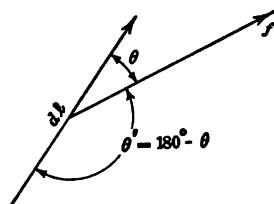


FIG. 7.

where $\theta$ is the angle between the direction of the force and the direction of the displacement of the particle, as shown in Fig. 7. In this definition the displacement $dl$ is the displacement of the particle on which the force acts, *relative* to the agent which produces the force.

When a particle moves along any path of length $l$, straight or curved, and at each point of this path is acted upon by a force, of constant or variable magnitude and direction, the total work done on it is

$$W = \int_o^l (f \cos \theta) \, dl \qquad\qquad (19a)$$

where the integral sign indicates the summation of the products $(f \cos \theta) \, dl$ for all the infinitesimal lengths into which the path may be assumed to be divided. An integral of this form is called a "line integral." Hence the work done on a moving particle by a force is the line integral of the force along the path over which the particle moves.

From the definition of the work done on a moving particle by a force, it follows that when the direction of motion of the particle makes an angle greater than 90 degrees with the direction of the force, then the work done by the force is negative. The particle is then said to do a numerically equal amount of work *against* the force, or *on* the agent which produces the force. Or, calling $\theta'$ the angle between the direction of the force $f$ and the direction opposite to that of the displacement $dl$, the work done against the force is

$$dW = (f \cos \theta') \, dl \qquad\qquad (19b)$$

From the definition it follows that work is a scalar quantity, *i.e.*, the amount of work done by a force may be either positive or negative, but it has no direction in space.

When work is done on a body, a change in its position, velocity or in some other property is always produced. Certain properties of a body which can be altered by doing work on it can also be altered by other means. For example, when two bodies are rubbed together rapidly, mechanical work is done on them by the agent which produces the force which moves one over the other against the opposing force due to "friction." As a result, the temperature of the bodies is raised; but the temperature of a body may also be raised by placing it near or in contact with a hotter body, even though no work is required to place it in this position.

In general, any change produced in a body or system of bodies by any means whatever, which change can also be produced directly by doing work on that body or system, or indirectly by doing work on some other body or system, is said to be due to a transfer of "energy" to that body or system. As a measure of the gain in energy corresponding to this change is taken the amount of work which would have to be done on it to produce this change and no other.

Conversely, whenever a change takes place in a body or system of bodies which is the reverse of the change which can be produced in it by doing work on it, the body or system is said to lose energy, or energy is said to be transferred from it. As a measure of the loss of energy corresponding to this change is taken the amount of work which would have to be done on it to produce the reverse of this change and no other.

Experience indicates that the amount of energy which can be

transferred from a body or system of bodies to which no energy is added, is limited, *i.e.*, the energy "possessed by," or "associated with," any body or system of bodies is finite in amount. Therefore the amount of work which can be done by a given body or system of bodies is never greater than the energy possessed by this body or system. Hence, the energy of a body or system of bodies may be defined qualitatively as the "capacity of this body or system for doing work."

Experience also justifies the assumption that whenever one body or system of bodies gains energy, some other body or system loses an exactly equal amount of energy. In every instance where this assumption can be tested directly, it is found to hold, and every deduction from it has been found to be in accord with experimental fact. Hence, this assumption is accepted as a fundamental principle of nature. This principle, which is equivalent to the statement that energy is never created or destroyed, but can only be transferred or transformed, is known as the "principle of the conservation of energy."

It is found convenient, in discussing the various properties of matter, to look upon each property of a portion of matter as corresponding to a definite amount of energy, and to give a special name to the energy associated with each property. For example, the energy associated with a body in virtue of its velocity is called its kinetic energy; the energy associated with a body in virtue of its position with respect to other bodies which exert forces on it is called its potential energy; the energy possessed by a body in virtue of its temperature is called its heat energy, or its thermal energy; the energy associated with a body in virtue of its chemical nature is called its chemical energy, etc.

**16. Units of Work and Energy.**—The fundamental absolute unit of mechanical work in the c.g.s. system is the work done by a force of 1 dyne when its point of application is displaced (with respect to the agent producing the force) a distance of 1 centimeter in the direction of the force; this unit is called the "erg." $10^7$ ergs, namely, 10,000,000 ergs, is called a "joule." As an example of a c.g.s. gravitational unit may be cited the kilogram-meter, which is the work required to raise 1 kilogram 1 meter at a place where $g_0 = 980.665$ centimeters per second per second.

The fundamental unit of mechanical work in the English gravitational system is the foot-pound, which is the work done by a

force of 1 pound when its point of application is displaced a distance of 1 foot in the direction of the force; or 1 foot-pound is the work required to raise a mass of 1 pound a distance of 1 foot at a place where the acceleration due to gravity is 32.1739 feet per second per second. 1,980,000 foot-pounds is called a horse-power-hour (see Article 19).

Energy is expressed in the same units as mechanical work. In addition other units of energy are employed for particular forms of energy, such as the calorie and British thermal unit for heat energy, the kilowatt-hour for electric energy, etc. The mean gram-calorie is defined as the one-hundredth part of the heat required to raise the temperature of 1 gram of water from 0° to 100°C. The British thermal unit, as defined by Marks and Davis, is one one-hundred-eightieth part of the energy required to raise the temperature of 1 pound of water from 32° to 212° F. The kilowatt-hour is defined in Article 19. Each of these units bears a definite, fixed relation to each of the others; see any engineering handbook.

**Problem 9.**—Referring to Problem 7:

(a) How much work is done by the locomotive in hauling the train at a constant speed a distance of 1 mile up the given grade? (b) If the train accelerates throughout this distance at the rate of 0.5 miles per hour per second, how much work will the locomotive do?

*Answer.*—(a) 149 horsepower-hours. (b) 247 horsepower-hours.

**17. Kinetic and Potential Energy.**—It can readily be deduced from the definitions given in the preceding articles that the energy possessed by a particle in virtue of its motion, *i.e.*, its kinetic energy, is $\frac{1}{2} mv^2$, where $m$ is its mass and $v$ its velocity. This expression gives the kinetic energy of the particle in *absolute* units. The corresponding expression for the kinetic energy of a moving particle in *gravitational* units is $\frac{1}{2} \frac{m}{g_o} v^2$.

The kinetic energy of a rigid body which is rotating with an angular velocity of $\omega$ radians per second about a fixed axis is $\frac{1}{2} I\omega^2$ absolute units, or $\frac{1}{2} \frac{I}{g_o} \omega^2$ gravitational units, where $I$ is the moment of inertia of the body about this axis.

When the center of mass of a rigid body of mass $M$ is moving with a linear velocity of $v$, and at the same time this body is rotating with an angular velocity $\omega$ radians per second about an axis

which passes through its center of mass and which moves with the body, the total kinetic energy of the body is

$$\frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2 \qquad \text{absolute units} \qquad (20)$$

or

$$\frac{1}{2} \frac{M}{g_o} v^2 + \frac{1}{2} \frac{I}{g_o} \omega^2 \qquad \text{gravitational units} \qquad (20a)$$

where $I$ is the moment of inertia of the body about this axis. A case of this kind is the wheel on a moving car, or the armature of the motor which drives the wheel.

The above formulas apply only when all the quantities entering into the particular formula used are expressed in the same system of units. For example, to calculate the energy in ergs, the mass must be expressed in grams, the linear velocity in centimeters per second, the angular velocity in radians per second, and the moment of inertia in grams and centimeters. Again, to calculate the energy in foot-pounds, the mass must be expressed in pounds, the linear velocity in feet per second, the angular velocity in radians per second, the moment of inertia in pounds and feet, and $g_o$ must be taken equal to 32.1739.

The potential energy possessed by a particle of mass $m$ in virtue of its elevation with respect to the surface of the earth is $mgh$ absolute units, or $\dfrac{mgh}{g_o}$ gravitational units, where $h$ is its elevation above the surface of the earth, $g$ is the acceleration due to gravity at the place in question, and $g_o$ is a number equal to the value of the standard acceleration due to gravity in the system of units employed (see Article 11).

Problem 10.—(a) What is the kinetic energy in foot-pounds of a mass of 1 ton (2000 pounds) moving with a linear velocity of 60 miles per hour? (b) If a retarding force of 150 pounds is applied to this mass, how many seconds will be required to bring it to rest? (c) How many feet will it travel before coming to rest? (d) An 800-ton train is traveling on a straight level track at a velocity of 60 miles an hour when the brakes are applied. The retarding force produced by the brakes is 120,000 pounds. How many seconds will be required to bring the train to rest? (e) How many feet will it travel before coming to rest? (Neglect the kinetic energy of rotation of the wheels.)

Answer.—(a) 241,000 foot-pounds. (b) 36.5 seconds. (c) 1606 feet. (d) 36.5 seconds. (e) 1606 feet. (Note.—These figures are typical of actual conditions in railway practice.)

**Problem 11.**—If the cylinder in Problem 5 is rotating at a speed of 1000 r.p.m., what is its kinetic energy: (a) In foot-pounds? (b) In horse-power-hours?

*Answer.*—(a) 10,090,000 foot-pounds. (b) 5.10 horsepower-hours.

**Problem 12.**—If the force driving the cylinder in Problem 11 is removed, how many hours will it take the cylinder to come to rest, assuming the opposing torque due to friction to be constant and equal to 30 pound-feet?

*Answer.*—1.78 hours. (Note.—The data given in Problems 11 and 12 are fairly representative of a large turbo-generator, except that the friction torque is not constant. Unless a brake of some kind is used, the rotating part of such a machine may continue in motion for several hours after the steam is shut off.)

**Problem 13.**—A weight of 500 pounds falls from rest a distance of 100 feet into a tub of water. If the tub contains 10 gallons of water, and all the kinetic energy lost by the weight when it is brought to rest goes into heating the water, what will be the rise in the temperature of the water produced thereby, assuming no radiation of heat energy? Give answer in degrees centigrade and degrees Fahrenheit.

*Answer.*—0.427°C. = 0.770°F.

**18. Frictional and Elastic Forces.**—It is a matter of experience that work is always required to move one portion of matter relative to another with which it is in contact, and that as a result of this motion heat energy is developed within the two bodies. This fact is conveniently described by the statement that between two bodies in contact there exists a "frictional force" which opposes the relative motion of the two bodies.

As the measure of this frictional force is taken the force which must be applied to "overcome" it, *i.e.*, that force which will do an amount of work equal to the heat energy developed, when the displacement of the point of application of this force is equal to the relative displacement of the two bodies. Hence, if $f$ is the frictional force, and $dl$ the relative displacement of the two bodies, then the corresponding amount of work done against friction ( = the heat energy developed) is $dW = fdl$.

The frictional force between two portions of matter is found to depend upon the nature of the two bodies, the extent and condition of the two surfaces in contact, the pressure normal to the surface of contact, the velocity of the relative motion between them, etc.

Experiment also shows that in general work must be done on a portion of matter in order to change either its size or shape, and, moreover, that a force is required to maintain it in any size or

shape other than that which it assumes when no force is exerted upon it. When a body is deformed in any manner it is said to exert a reactive force due to its "elasticity," or to exert an "elastic force."

The work done against an elastic force does not appear as heat energy, but when the force which does the work is removed, the body by returning to its original size or shape can do work on, or give out energy to, some other body. Hence, from the principle of the conservation of energy, the work done against elastic forces must be looked upon as converted into a form of energy which is stored in the body. For example, when a spiral spring is stretched, energy is stored in the spring, for the stretched spring has the capacity for doing work.

As the measure of an elastic force is taken the force which must be applied to balance it, *i.e.*, to keep it from producing any acceleration of the common point of application of these two forces.

The elastic force exerted when a body is deformed in any manner is found to depend upon the nature of the body, its size and shape, and the nature of the deformation. When the deformation is expressed in a suitable manner, the general law connecting the elastic force and the deformation which it produces is that the elastic force is proportional to the deformation.

**Problem 14.**—A weight of 4 pounds is suspended from one end of a spiral spring of negligible mass, and the other end of this spring is fastened to a rigid support. A second weight of 1 pound is hung from the first weight by a string, causing the center of mass of the 4-pound weight to be displaced vertically downward 0.5 inch from its equilibrium position, which latter may be designated as the point $E$.

(a) If the string is burnt off, by setting fire to it with a match, thereby letting the 1-pound weight drop, what will happen to the 4-pound weight? (b) What is the value and direction of the resultant force acting on the 4-pound weight at the instant at which the 1-pound weight drops? (c) What will be the acceleration of the first weight, in inches per second per second, at any instant $t$ seconds after the second weight falls, assuming (as is practically the case) that the resultant force acting on it due to the tension of the spring and gravity is proportional at each instant to the distance of the center of mass of the 4-pound weight from its equilibrium position $E$? (d) What will be its velocity at this instant, in inches per second? (e) What will be its distance, at this instant, in inches, from its equilibrium position $E$? (f) How many complete oscillations will the weight make per second? (g) What is the total kinetic and potential energy at any instant, with respect to the equilibrium position $E$, of the system formed by the weight and

spring. (*h*) Using time as abscissa, plot to the same origin, the acceleration, velocity and displacement of this weight. (In the above questions all friction is to be neglected.)

*Answer.*—(*a*) The weight will oscillate vertically about its equilibrium position. (*b*) 1-pound vertically upward toward the equilibrium position *O*. (*c*) The force acting toward *E* when the center of mass of the 4-pound weight is at a distance of *x* inches from *E* is 2*x* pounds, since a force of 1 pound corresponds to a displacement of 0.5 inch. Hence for a displacement of *x* inches, the acceleration of the 4-pound weight is $\dfrac{2x \times 32.17 \times 12}{4} =$ 193*x* inches per second per second in the *opposite* direction to the displacement *x*. Therefore, the acceleration toward *E* at any instant is

$$\frac{d^2x}{dt^2} = -193x$$

The solution of a differential equation of this form is

$$x = A \sin (\sqrt{193}\,t + \theta)$$

where *A* and *θ* are constants fixed by the initial conditions. Since in this particular problem *x* = 0.5 when *t* = 0, and the velocity of the weight, namely $\dfrac{dx}{dt}$, is also zero at time *t* = 0, therefore *A* = 0.5 and $\theta = \dfrac{\pi}{2}$. Whence the acceleration at time *t* is 96.5 cos (13.9*t*) inches per second per second, where (13.9*t*) = $\sqrt{193}\,t$ is in radians. (The corresponding angle in degrees is 796*t*.) (*d*) −6.95 sin (13.9*t*) inches per second downward. (*e*) 0.5 cos (13.9*t*) inches downward. (*f*) 2.21 complete oscillations or "cycles" per second. (*g*) ⅙ foot-pound at all instants, the potential energy decreasing as the kinetic energy increases.

**19. Power.**—The rate, with respect to time, at which work is done, or at which energy is transferred or transformed, is called "power." For example, when a weight of 10 pounds is lifted a distance of 5 feet, 50 foot-pounds of work are done on this weight. If the weight is raised this distance in 2 seconds at a uniform speed, then the power at each instant is 50 ÷ 2 = 25 foot-pounds per second.

In general, when *W* units of energy are transferred at a uniform rate to a body or device of any kind, and the total time required for this transfer of energy is *t*, then the power at each instant during this interval is

$$P = \frac{W}{t} \tag{21}$$

When the rate of transfer is not uniform then the power at any instant is

$$p = \frac{dW}{dt} \qquad (21a)$$

where $dW$ is the energy transferred during an infinitesimal interval $dt$ measured from this instant.

Conversely, when the power is constant over an interval $t$, the corresponding energy transfer during this interval is

$$W = Pt \qquad (21b)$$

When the power varies during the interval $t$, the corresponding energy transfer is

$$W = \int_o^t p\,dt \qquad (21c)$$

Although power is strictly the rate of doing work, and a rate cannot be transferred, yet it is common usage to speak of the power input or output of a device, meaning thereby the rate at which energy is transferred to it or from it respectively.

The fundamental absolute unit of power in the c.g.s. system is 1 erg per second. This unit is extremely small, and is seldom used. A more convenient unit is the joule ($= 10^7$ ergs) per second. This unit is called the "watt," from which is derived the "kilowatt," which by definition is 1000 watts.

In the English gravitational system of units the fundamental unit of power is the "horsepower," which is the uniform rate at which work must be done to raise 1 pound 550 feet in 1 second, *i.e.*, 1 horsepower is equal to 550 foot-pounds per second. For the interrelations of the various units of power see any engineers' handbook.

From the kilowatt is derived the energy unit called the "kilowatt-hour," which may be defined as the energy transferred in 1 hour by a constant power of 1 kilowatt (see also Article 36)

Since work is force times distance, and power is the rate of doing work, it follows that power is also equal to force times velocity, since velocity is the rate of change of distance. That is, the power corresponding to a force $f$ when the point of application of this force is moving with a velocity $v$, is

$$p = fv \qquad (22)$$

When the point of application of a force $f$ moves in a circle of radius $r$ at a velocity $v$, the angular velocity of this point is $\omega = \dfrac{v}{r}$,

and the torque corresponding to the force $f$ is $T = rf$. Hence power is also equal to the product of torque by angular velocity, viz.,

$$p = T\omega \tag{22a}$$

In both of these equations all quantities must be expressed in the same system of units, and in particular, in equation (22a), $\omega$ must be expressed in radians per unit of time. For example, if the force in equation (22) is in pounds and the velocity in feet per second, then the power is in foot-pounds per second; the corresponding horsepower is then $\dfrac{fv}{550}$.

It is often convenient to express angular velocity in revolutions per minute. Since one revolution per minute is equal to 60 revolutions per second, and since one revolution equals $2\pi$ radians, the horsepower corsponding to a torque of $T$ pound-feet and $N$ revolutions per minute is

$$p = \frac{2\pi}{60 \times 550} NT = \frac{NT}{5252} \tag{22b}$$

**Problem 15.**—Referring to Problem 8, how many horsepower are transferred through the pulley?

*Answer.*—9.52 horsepower.

**Problem 16.**—Referring to Problem 7:

(a) If the speed of the train moving up the 3 per cent. grade is constant at 30 miles per hour, how many horsepower must the locomotive develop? (b) If the speed at a given instant is 30 miles per hour and the train is accelerating at this instant at the rate of 0.5 miles per hour per second, how many horsepower must the locomotive develop? (c) If the speed at a given instant is 30 miles per hour and the train at this instant is slowing down at the rate of 0.5 miles per hour per second, how many horsepower must the locomotive develop?

*Answer.*—(a) 4480 horsepower. (b) 7410 horsepower. (c) 1530 horsepower.

**20. Efficiency and Losses.**—In any machine or device which is employed for transforming energy from one form into another, or for transferring energy from one place to another, a certain amount of energy is always converted within the device into forms which cannot be readily utilized. In general, this useless energy appears as heat energy.

The difference between the power input and the useful power output is called the power loss. The ratio of the useful output $P_o$ to the input $P_i$ for any given "load" on a machine is defined

as the efficiency of the machine at this load. This ratio is usually expressed as a percentage. The per cent. efficiency $\eta$ is then

$$\eta = 100 \frac{P_o}{P_i} \qquad (23)$$

The ratio of the difference between the input $P_i$ and the output $P_o$ to the input $P_i$, expressed as a percentage, is called the per cent. power loss $q$; that is,

$$q = 100 \frac{P_i - P_o}{P_i} = 100 - \eta \qquad (23a)$$

Sometimes it is more convenient to express the power loss as a percentage of the output $P_o$. Let $q'$ be the percentage loss expressed in this manner; then

$$q' = 100 \frac{P_i - P_o}{P_o} \qquad (23b)$$

Whence

$$q = \frac{q'}{1 + 0.01\, q'} \qquad (23c)$$

$$\eta = \frac{100}{1 + 0.01\, q'} \qquad (23d)$$

The efficiency of a machine varies as a rule with the load or output. For no load, *i.e.*, no useful output, the efficiency is zero, since in general energy must be supplied to the machine to operate it, even though the machine does no useful work. As the load comes on, the efficiency increases up to a certain output, depending on the design of the machine, and then decreases.

The "rated load," or rated power output, of a machine which is designed for continuous service, is the maximum rate at which useful energy may be transferred through the machine continuously without injury to any of its parts. In most electric machinery this is determined by the rise of temperature produced by the energy dissipated, or "lost," in the windings and iron cores. For machines in which fibrous insulation (such as cotton or paper) is used, the insulation will as rule, begin to deteriorate if its temperature exceeds 105°C.

Problem 17.—The thermal efficiency of a given boiler is 65 per cent. (*i.e.*, of the total energy in the coal 65 per cent. can be transferred to the steam). The efficiency of a given steam engine at full load is 20 per cent. (*i.e.*, of the total energy in the steam which passes through the engine only 20 per cent.

is converted into mechanical energy). A pound of high-grade coal contains 14,000 British thermal units of energy.

(a) What is the overall efficiency of boiler and engine? (b) How many pounds of coal will be required to produce 1 horsepower-hour at the engine shaft?

*Answer.*—(a) 13.0 per cent. (b) 1.40 pounds of coal per horsepower-hour. (For average load conditions, due to the reduced efficiency at partial loads, the coal consumption per horsepower-hour is in practice considerably greater than this, ranging from about 2 pounds per horsepower in plants of 20,000 horsepower or more to about 5 pounds per horsepower-hour in small plants.)

**Problem 18.**—(a) If the efficiency of a water wheel is 80 per cent., how many cubic feet of water per second falling through a distance of 1 foot will be required to develop 1 horsepower? (b) Let $Q$ be the number of cubic feet per second, $H$ the "head" or distance through which the water falls, and $P$ the horsepower; what is the relation between $P$, $Q$, and $H$ for a wheel of 80 per cent. efficiency?

*Answer.*—(a) 11 cubic feet per second. (b) $P = \dfrac{QH}{11}$.

**21. Flow of Water Through Hydraulic Motors and Pumps.**— Referring to Fig. 8, $M$ represents a hydraulic motor (of the turbine, or rotating vane, type) with its intake at $A$ and its outlet at $B$. To force water through the motor in the direction indicated by the arrow marked $I$, the pressure at the intake $A$ must be greater than that at the outlet $B$, as indicated by the difference in the water levels in the two small open vertical tubes.

Let $V_A$ and $V_B$ be the absolute pressures, in pounds per square foot of cross-sectional area of the stream of water, at the intake $A$ and outlet $B$ respectively. Let the intake and outlet pipes have the same cross-section, namely, $S$ square feet. Then the water coming up to the intake $A$ exerts on the mass of water in the motor between $A$ and $B$ a force of $V_A S$ pounds, in the direction of the flow, and the water in the external pipe at the outlet $B$ exerts on this same mass a force of $V_B S$ pounds, in the direction opposite to that of the flow. Hence the net, or resultant, driving force acting on the mass of water between $A$ and $B$ is $(V_A - V_B)S$ pounds, in the direction of the flow.

If the velocity of the stream of water from $A$ to $B$ remains constant, then the resultant of the opposing forces acting on this mass of water, due to friction in the water passages of the motor and to the reaction of the vanes of the motor on the water which drives them, must be equal to the driving force $(V_A - V_B)S$. Let $F_r$ be the frictional force, in pounds, acting on the mass of

water between $A$ and $B$, and let $F$ be the opposing force, in pounds, exerted on this water by the vanes of the motor. ($F$ is numerically equal to the driving force exerted by the water on these vanes.) The resultant opposing force is then $F_r + F$ pounds, which, equated to the resultant driving force, gives

$$(V_A - V_B)S = F + F_r$$

The work, in foot-pounds, done by the resultant force $(V_A - V_B)S$ when 1 cubic foot of water flows past $A$, and therefore also past $B$, is equal to the product of this force times the corresponding linear displacement, in feet, of the mass of water originally between $A$ and $B$. This displacement is $\frac{1}{S}$, where $S$ is the cross-section, in square feet, of the pipe at $A$ or $B$. Hence the foot-
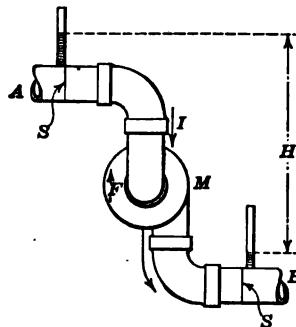


Fig. 8.—Hydraulic motor.

pounds of work done on the water between $A$ and $B$ per cubic foot of water forced through the motor is $(V_A - B_B)S \times \frac{1}{S} = V_A - V_B$.

Hence, *the drop in pressure from the intake to the outlet of a hydraulic motor is equal to the work done on the water which is forced through it per unit quantity of water which enters its intake or leaves its outlet.* It can be shown, as a consequence of the principles stated in the preceding articles, that this relation is a perfectly general one, irrespective of the relative cross-sections of the intake and outlet and of the constancy of the velocity of the stream of water.

The drop in pressure $(V_A - V_B)$ is proportional to the difference in the elevation of the tops of the water columns in the two

vertical tubes, being equal, in pounds per square foot, to 62.5 $H$, where $H$ is this difference in elevation in feet, and 62.5 is the weight, in pounds, of 1 cubic foot of water (at average atmospheric temperature).

The above relation between the driving force and the opposing forces may also be written

$$V_A - V_B = \frac{F}{S} + \frac{F_r}{S}$$

The term $\frac{F_r}{S}$ may be designated the "drop of pressure due to the resistance" of the water passages of the motor, and $\frac{F}{S}$ may be designated the "back pressure" exerted by the vanes of the motor. Let $E$ be this back pressure developed by the vanes of the motor, *i.e.*, put $E = \frac{F}{S}$, and let $V_r$ be the internal resistance drop, *i.e.*, put $V_r = \frac{F_r}{S}$. Then the relation just stated may be written

$$V = E + V_r \tag{24}$$

where $V$ is the difference in pressure between intake and outlet.

The work done by the water against the frictional force $F_r$ produces heat energy. Hence in this equation $V_r$ is numerically equal to the heat energy developed within the motor per cubic foot of water forced through it. Similarly, $E$ is numerically equal to the work done *by* the water *on* the moving vanes of the motor per cubic foot of water forced through it, and $V$ is the total work done by the external driving force per cubic foot of water forced through the motor.

Let $I$ be the number of cubic feet of water which is forced through the motor per second. Then the number of foot-pounds of work per second required to force this stream or "current" of water through the motor is $VI$. That is, the power input to the motor is $P_i = VI$ foot-pounds per second.

Similarly, $V_rI$ is equal to the total power which is dissipated within the motor as heat, and $EI$ is the power transferred from the water to the vanes of the motor, both in foot-pounds per second. Whence, equation (24) is equivalent to the relation

$$P_i = VI = EI + V_rI \tag{24a}$$

which is merely a convenient way of stating the fact that the

total work per second done *on* the water which is forced through
the motor is equal to the work per second done *by* the water on
the moving vanes of the motor, *plus* the heat energy developed
per second within the motor between its intake and outlet.

Note that $EI$ is greater than the power available at the shaft
of the motor by the power which is lost due to the friction in the
shaft bearings. That is, $V_rI$ in equation (24a) includes only
the power lost due to friction in the water passages of the motor,
and not that due to journal friction.

In an exactly similar manner it may be shown that the *rise
of pressure* from the intake $A$ to the outlet $B$ of a centrifugal
pump $G$, see Fig. 9, is equal to the work done by the pump on the
water in the external pipe or pipes connected to its intake and
outlet, and that this rise of pressure is, in turn, equal to the
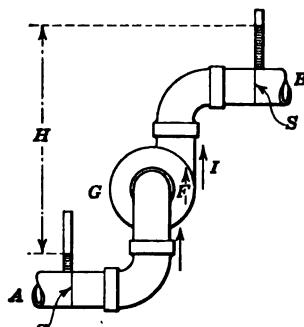


FIG. 9.—Hydraulic pump.

driving pressure developed by the pump, *less* the back pressure
caused by the frictional resistance which opposes the flow of
water through the pump.

This relation may be expressed in symbols as follows. Let $V$
be the rise of pressure, in pounds per square foot, from the intake
$A$ to the outlet $B$ of the pump $G$ in Fig. 9. As before, $V = 62.5H$,
where $H$ is the difference in elevation, in feet, between the water
levels in the gages at the intake and outlet of the pump. Let $S$
be the cross-sectional area, in square feet, of the intake and outlet
pipes, let $F_r$ be the total frictional force, in pounds, within the
pump itself, opposing the flow of water through it, and let $F$
be the total driving force, in pounds, exerted on the water by the
vanes of the pump. As before, the pressure $\dfrac{F_r}{S}$ may be desig-

nated the drop of pressure due to the frictional resistance of the water passages of the pump, and $\dfrac{F}{S}$ the total driving pressure developed by the pump. Representing these two pressures by $V_r$ and $E$ respectively, then

$$V = E - V_r \qquad (25)$$

The net power output of the pump, *i.e.*, the foot-pounds of work done per second by the pump on the water in the external pipe or pipes connected to its outlet and intake, when the flow of water through it is at the rate of $I$ cubic feet per second, is then

$$P_o = VI = EI - V_rI \qquad (25a)$$

which is merely a convenient way of stating the fact that the external work done per second by a pump is less than the work done per second by the driving vanes, by an amount equal to the heat energy developed per second within the pump due to the frictional resistance of its water passages.

The reader should compare carefully the *motor* equations, (24) and (24a), with the *pump* equations (25) and (25a), and assure himself that he understands thoroughly the significance of the symbols used and the physical facts which these relations express. As shown in the next chapter, relations of identically the same forms hold for the flow of electricity through electric motors and generators, and, in fact, for the flow of electricity through any kind of a device or circuit.

**Problem 19.**—The difference in pressure between the intake and outlet of a centrifugal hydraulic motor is 2000 pounds per square foot. The motor develops at its shaft 10 horsepower, or 5500 foot-pounds of energy per second, and 1000 foot-pounds of energy per second are dissipated as heat due to the frictional resistance of the water passages through the motor. Assume that the intake and outlet have the same cross-section and that the friction of bearings supporting the rotating vanes is negligible.

(a) What is the total power input to the motor? (b) What is the total energy input per cubic foot of water forced through it? (c) What is the total power input per cubic foot of water per second? (d) What must be the rate of flow of water through the motor? (e) What is the back pressure developed by the vanes of the motor per square foot of the area of the intake or outlet? (f) What is the energy output of the motor per cubic foot of water forced through it? (g) What is the power output of the motor per cubic foot of water per second? (h) What is the efficiency of the motor? (i) What is the direction of the flow of water through the motor with respect to the drop of pressure through it?

*Answer.*—(a) 6500 foot-pounds per second. (b) 2000 foot-pounds per cubic foot. (c) 2000 foot-pounds per second per cubic foot per second. (d) 3.25 cubic feet per second. (e) 1692 pounds per square foot. (f) 1692 foot-pounds per cubic foot. (g) 1692 foot-pounds per second per cubic foot per second. (h) 84.6 per cent. (i) The drop in pressure and flow are in the same direction.

**Problem 20.**—The difference in pressure between the intake and outlet of a centrifugal pump is 2000 pounds per square foot. 10 horsepower, or 5500 foot-pounds of energy per second, are required to drive the rotating vanes, and 1000 foot-pounds of energy per second are dissipated as heat due to the frictional resistance of the water passages of the pump. Assume that the intake and outlet have the same cross-section and that the friction of the bearings supporting the rotating vanes is negligible.

(a) What is the total power output of the pump? (b) What is the net energy output per cubic foot of water pumped? (c) What is the net power output per cubic foot of water per second? (d) At what rate does the pump take in and discharge water? (e) What is the pressure exerted on the water by the rotating vanes per square foot of the intake or outlet? (f) What is the energy input to the pump per cubic foot of water pumped? (g) What is the power input to the pump per cubic foot of water per second? (h) What is the efficiency of the pump? (i) What is the direction of the flow of water through the pump with respect to the drop of pressure through it?

*Answer.*—(a) 4500 foot-pounds per second. (b) 2000 foot-pounds per cubic foot. (c) 2000 foot-pounds per second per cubic foot per second. (d) 2.25 cubic feet per second. (e) 2444 pounds per square foot. (f) 2444 foot-pounds per cubic foot. (g) 2444 foot-pounds per second per cubic foot per second. (h) 81.8 per cent. (i) The drop in pressure through the pump is opposite to the flow of water through it; *i.e.*, the flow is in the direction of the rise of pressure.

# II

# ELECTRICITY, ELECTRIC CURRENTS AND ELECTRIC ENERGY

**22. Introduction.**—In the following pages the attempt is made to give, in as brief a space as is consistent with clearness: (1) a description of the more important effects commonly described as electric and magnetic phenomena, (2) a statement of the fundamental laws in accord with which these phenomena have been found, by observation and experiment, to occur, and (3) the applications of these laws to some of the simpler problems which arise in connection with the generation, transmission and utilization of electric energy.

Particular emphasis will be laid upon exact *quantitative* statements of the fundamental laws or principles. Both safety and economy demand that the engineer be able to answer not only "how," but also "how much." In short, the engineer must be able to analyze, both qualitatively and *quantitatively*, each problem which may be presented to him.

Most of the simpler formulas used by scientists and engineers are special cases of certain general relations, and these special formulas are applicable only under certain specific conditions. One of the most common causes of confusion on the part of the beginner arises from his attempt to apply such special formulas to cases to which they are not applicable. This is due in part to the failure in many text-books to state the *limitations* of such formulas. Particular care will therefore be taken in the following chapters to state specifically the exact conditions under which each formula is applicable.

The procedure adopted in the following chapters is to pass from simple phenomena known to practically every school-boy to the more complex phenomena and principles with which the engineer has to deal.

**23. Voltaic Cells.**—Any two dissimilar metals, or carbon and a metal, partly or wholly immersed in an acid or salt solution, consti-

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tute what is known as a "voltaic cell." The two pieces of metal (or metal and carbon) are usually referred to as the "plates," or "poles," of the cell, and the solution, for reasons explained in Article 27, is called the "electrolyte." In any practical form of such a cell the two plates are kept from touching each other within the container, and each is provided with a suitable
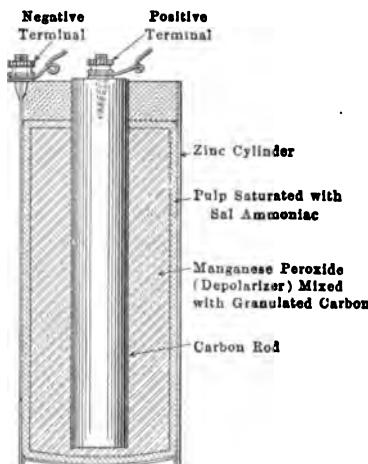


FIG. 10.—Cross-section of a dry cell.

"terminal," or "binding-post," to which one or more wires may be connected.

The commonest form of voltaic cell in use at the present time is the so-called "dry cell," or "dry battery." Such a cell is not actually dry inside, but contains a layer of absorbent material saturated with a suitable electrolyte. The container, usually a
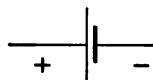


FIG. 11.

hollow zinc cylinder, forms one of the "plates," the other "plate" being a carbon rod. Fig. 10 shows a cross-section of such a cell.

The carbon plate of a dry cell is called its "positive" plate, and the terminal connected thereto its "positive" terminal, and the zinc plate and terminal connected thereto the "negative" plate and "negative" terminal respectively. The conventional diagram of a cell of any kind is shown in Fig. 11, the long line

representing the positive and the short line the negative plate. When several cells are connected in series, *i.e.*, with the positive terminal of one to the negative terminal of the next; or in parallel, *i.e.*, with like terminals all connected, they are said to form a "battery" of cells. A single cell is also frequently called a battery.

For a detailed description of the construction of various types of voltaic cells, the student is referred to any electrical engineers' handbook.[1] Every student of electrical engineering should own such a book, and should get into the habit of consulting it, particularly for numerical data and other information in regard to engineering practice.

**24. Electricity and Magnetism.**—Take a piece of insulated wire (*e.g.*, a wire covered with cotton or silk) and wind it around a piece of iron or steel (*e.g.*, a nail or bolt), and connect the free ends of the coil thus formed to the two terminals of a dry cell. Two distinct phenomena are then readily observed, viz.:

1. The wire becomes heated.

2. The iron acquires the power of attracting bits of iron or steel.

These two effects, as well as others to be described later, are attributed to the "flow of electricity" through the wire, and the new property acquired by the iron is said to be due to its becoming "magnetized" under the influence of this flow of electricity. Note that the flow of electricity is through the wire which surrounds the iron, *not* through the iron. A flow of electricity is commonly called an "electric current," and the condition in the region surrounding the path, or "circuit" of this flow, in virtue of which a piece of iron placed in this region becomes magnetized, is said to be due to the establishment by the current of a "magnetic field."

An inspection of the active materials of the cell, after it has established for some time a flow of electricity through any device connected to its terminals, will show that this flow is accompanied by a chemical change within the cell, and that this change is of such a nature that the cell *loses energy*. That is, to restore the

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[1] In this country there are three comprehensive electrical engineers' handbooks on the market, viz., PENDER'S HANDBOOK FOR ELECTRICAL ENGINEERS (John Wiley and Sons), the STANDARD HANDBOOK FOR ELECTRICAL ENGINEERS (McGraw-Hill Book Co.), and FOSTER'S ELECTRICAL ENGINEERS' POCKET BOOK (D. Van Nostrand Co.).

active materials of the cell to their original condition, energy from some other source must be supplied to it.

The heating of the wire is a manifestation of energy developed within it. From the principle of the conservation of energy, it is perfectly rational to look upon the energy which is lost by the cell as being transferred from the cell to the wire, and to look upon the something called electricity as the means whereby this transfer of energy takes place.

Moreover, since the cell loses energy (*i.e.*, does work), it may be looked upon as the cause of the flow of electricity through the "circuit" formed by itself and the wire. This is commonly described by saying that the cell is a source of an electricity-moving force, or briefly, of an "electromotive force."

Similarly, any device which is capable of causing, in and around a wire connected to its terminals, the two effects above described, is said to be the source of an electromotive force. A device which produces a flow of electricity is analogous to a pump, and the electromotive force of the cell is analogous to the pressure developed by the pump (see Article 21).

A voltaic cell or other source of electromotive force is not to be looked upon as producing, or "generating," electricity, but merely as a device in which a force exists which is capable of setting electricity in motion. As will be explained in greater detail in the next article, electricity must be looked upon as something which exists normally in perfectly definite amounts in every piece of matter, but which may, under the action of certain forces, be caused to move from one portion of matter to another.

For example, in the simple apparatus above described, there exists both in the cell and in the wire a definite amount of electricity, even before the two are connected. When the wire is connected to the cell, the electromotive force of the cell merely sets this electricity in motion around the path, or circuit, formed by the cell and the wire, but does not create any new electricity. This motion continues as long as both ends of the wire remain connected to the terminals of the cell (provided the cell does not become "exhausted"). When either one, or both, of the ends of the wire are disconnected, this motion ceases; *i.e.*, the electric *current*, or *flow* of electricity disappears, and the electricity in the wire and cell returns to its normal state of equilibrium.

Another important group of electric phenomena is illustrated by the effects produced when two dry non-metallic bodies, such as glass and silk, are briskly rubbed together. Each of these two bodies acquires the property of attracting matter of any kind, *e.g.*, small bits of paper, and also of attracting or repelling other bodies which have been rubbed together. These effects are said to be due to the "production" of electric charges on the bodies in question. As will be pointed out later, the rubbing should not be looked upon as creating electricity, but rather as causing a transfer, from one body to the other, of the electricity which normally exists in each.

In the above discussion it is assumed, without proof, that electricity is an actual something. The justification for this assumption is that this hypothesis gives a rational basis for interpreting and correlating a large number of physical phenomena, and is found in no case to lead to an absurdity.

Magnetism, on the other hand, is not to be thought of as something possessing properties, but rather as itself a state or condition produced by electricity in motion. This is the modern view of magnetism, and is quite different from the old conception of "magnetic charges" as something located at the ends, or poles, of a magnet.

It therefore seems to the author more rational, although contrary to the historical development of the science, to take up first the various properties of electricity, and to treat magnetism as one of the properties of electricity in motion. Moreover, since the effects produced by electricity in motion can be measured more precisely than those due to electricity at rest, the properties of the electric current (*i.e.*, of moving electricity) will be considered first, and static electricity, or electricity at rest, will be discussed later.

**25. Electricity and Matter.**—Some of the simpler properties which observation and experiment indicate as necessary to assign to electricity are the following:

(*a*) Electricity exists in two distinct forms, which are conveniently designated as positive and negative electricity. Modern investigations have shown conclusively that one of these forms (the negative) is an actual entity which may exist independent of matter. The nature of the other form of electricity is not so certain, but until further researches have defi-

nitely established facts to the contrary, it may also be con-
sidered as an actual entity, which, however, is always associated
with matter.

(b) Every portion of matter in its normal or neutral state
contains equal quantities of the two kinds of electricity.

(c) Under the action of externally applied forces the two
kinds of electricity in a body may be displaced relatively to each
other, and any such force which tends to displace one form of
electricity in one direction will tend to displace the other form of
electricity in the opposite direction.

(d) The condition of a body described by stating that it is
positively charged may be looked upon as due either to the
addition of positive electricity to it, or to the withdrawal from
it of a definite quantity of negative electricity. In other words,
it is immaterial, as far as the ordinary phenomena of electricity
and magnetism are concerned, whether a positively charged body
is looked upon as having in addition to its normal amount of
negative electricity an *excess* of *positive* electricity, or whether it
is looked upon as having its normal amount of positive electricity
but a *deficit* of *negative* electricity.

Similarly, the condition of a body described by stating that
it is negatively charged may be looked upon as due either to the
addition of negative electricity to it, or to the withdrawal from
it of a definite quantity of positive electricity.

In order to fix definitely which kind of electricity shall be called
positive and which negative, physicists have agreed to call positive
the electric charge taken by glass when rubbed with silk.

(e) As a consequence of the above conception of electric
charges as being due to the transfer of one kind of electricity
from one body to another, it follows that the production of a
charge of one sign must always be accompanied by the production
somewhere else of an equal charge of the opposite sign, and that
the production of electric charges is always accompanied by the
motion or flow of electricity. For example, when glass is rubbed
with silk, and thereby given a positive charge, the silk acquires an
equal negative charge, due to the transfer of electricity from one
body to the other.

The practical unit of quantity of electricity is called the
"coulomb." The quantitative definition of a coulomb and of

the other units commonly employed for expressing quantity of electricity or electric charge will be given in a later section (Article 32).

(f) A further assumption in regard to the nature of electricity, which is in accord with all known facts, is that negative electricity exists in the form of discrete particles, each identical in every respect, irrespective of the chemical nature of the molecule with which it may be associated. These discrete particles of electricity are called "electrons."

**26. Conductors and Insulators.**—Since a flow of electricity always takes place through a wire when it is connected to the two terminals of a voltaic cell, the wire is said to be a "conductor" of electricity. When a dry cotton or silk string is connected to the terminals of the cell no appreciable flow of electricity takes place, *i.e.*, there is no appreciable heating of the string and no appreciable magnetic effect around it. Such a string is therefore called an "insulator" of electricity.

In general, any substance through which a flow of electricity can be produced, by connecting it to the terminals of a voltaic cell, is called an electric conductor. Similarly, any substance through which no readily detectable flow of electricity is produced when it is connected to the terminals of such a cell, is called an electric insulator. It should be noted, however, that there is no such thing as a *perfect* electric insulator; *i.e.*, every substance conducts electricity to at least a slight extent. However, the flow of electricity which a given source of electromotive force can produce, through dry, non-metallic solids (other than carbon) at ordinary temperatures, is practically inappreciable in comparison with that which this source of electromotive force can produce through metallic substances.

For most practical purposes, such substances as air and other gases at ordinary pressures, glass, porcelain, rubber, ebonite, gutta-percha, paraffine, silk, cellulose and shellac may be considered as insulators, while all metals, carbon, fused salts, solutions of most mineral salts and acids, and rarefied gases are conductors.

An insulator is frequently called a *dielectric*, particularly in the discussion of electrostatic effects (see Chapter XIII).

A conductor completely surrounded by insulators is said to be completely insulated. A wire is also said to be insulated when

it is completely covered with insulation except at its ends. That is, a rubber-covered wire, for example, with its ends connected to the poles of a voltaic cell, is spoken of as an "insulated wire connected to the cell."

**Problem 1.**—In Fig. 12 $A$, $B$ and $C$ represent three insulated conductors (*e.g.*, metal plates) and $B_1$, $B_2$ and $B_3$ represent three batteries of voltaic cells connected by wires to the three conductors through the contact-makers, or "switches," $K_1$, $K_2$, and $K_3$. Before the switches are closed each of the three conductors is in its neutral or normal state. When the switches are closed, 0.1 coulomb of negative electricity is transferred from $B$ to $A$, 0.06 coulomb of negative electricity from $A$ to $C$, and 0.075 coulomb of negative electricity from $C$ to $B$. Assuming that there is no transfer of positive electricity, what is the sign and amount of the charge on each conductor after these transfers have taken place?

*Answer.*—A negative charge of 0.04 coulomb on $A$, a positive charge of 0.025 coulomb on $B$, and a positive charge of 0.015 coulomb on $C$.
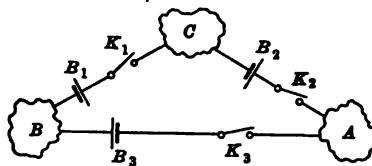


FIG. 12.

**27. Electrolytes and Electrolysis.**—When an electric current is established through the surface of contact between two metals, no appreciable chemical change is found to take place at this junction. However, when an electric current is established through the surface of contact between a metal and a solution of an inorganic salt or acid, a chemical change does in general take place at the metal-liquid junction. This may be readily shown by placing two copper plates in a solution of copper sulphate, and connecting these plates, with wires, to the two poles of a dry cell. The plate connected to the carbon terminal of the cell will waste away and the plate connected to the zinc terminal will have copper deposited on it.

Substances which, when placed in contact with a metal, undergo a chemical change when an electric current is passed through this surface of contact, are called "electrolytes," and the chemical action produced is designated as "electrolysis." Solutions of most inorganic salts and acids, certain fused salts, and a few solids are electrolytes.

As illustrated by the simple experiment just described, in order to establish an electric current in an electrolyte, it is necessary to have the electrolyte form part of a circuit containing a voltaic cell or other source of electromotive force. Metallic plates or rods dipping into the electrolyte and connected by wires to the terminals of the source of electromotive force serve to lead the current into and out of the electrolyte. These plates or rods are called "electrodes." The electrode at which the current enters is called the "anode," and that at which the current leaves the electrolyte is called the "cathode."

The chemical action in such an electrolytic cell originates only at the two electrodes. When the products of this decomposition are such that they can react chemically on the electrodes, the electrodes themselves undergo a chemical change, otherwise only the electrolyte is decomposed.

The constituents of an electrolyte which are primarily liberated or deposited at the electrodes are called "ions." An ion may be either an element, such as hydrogen or copper, or a chemical radical, such as $NO_3$ or OH. The ion liberated or deposited at the anode is called an "anion," and the ion liberated or deposited at the cathode is called a "cation." Hydrogen and metals are cations, and oxygen and the acid and base radicals, such as $NO_3$ and OH, are anions.

In any electrolytic action, the metal anode, if acted upon chemically by the liberated anion, is converted into a salt or base, and either goes into solution or is precipitated. At the cathode, on the other hand, the chemical action is a liberation of hydrogen when the electrolyte is an acid, or the liberation of a metal when the electrolyte is a salt or base. The metal may either be deposited on the electrode or may go into combination with the solvent in which the electrolyte is dissolved.

Electrolysis is the basis of a number of important industrial processes, as, for example, electroplating and electrolytic refining of copper and other metals. The electrolysis of iron pipes and other metallic structures buried in the earth (which is always more or less moist), due to electric currents which "leak" from the track rails of direct-current railway systems, sometimes causes serious corroding of these structures.

A further discussion of the phenomena of electrolysis is given in Chapter V.

**28. Quantity of Electricity and Current of Electricity.**—Whenever there is a flow of electricity in a circuit, during any interval of time a definite *quantity* of electricity flows through each cross-section of this circuit.   By the *rate* of flow of electricity through such a cross-section is meant the quantity of electricity which flows through this cross-section *per unit of time*.   This rate is called the "intensity," or "strength," of the electric current at this cross-section, or more commonly, simply the electric current at this cross-section.   That is, the term "electric current," when used in a quantitative sense, means the *time-rate of flow of electricity*.

The relation between quantity of electricity and current of electricity is exactly the same as that between the total number of cubic feet of water which flows through any given section of a pipe in any given time and the rate of flow of water through this section of pipe in cubic feet per second.

The intensity of an electric current is usually expressed numerically in terms of some magnetic effect which the current produces. Experiment shows that the magnetic effects produced by a positive charge moving in a given direction at a given speed are identical both in magnitude and direction with the like effects produced by an equal quantity of negative electricity moving at the same speed in the opposite direction.   (Compare also with the fact, paragraph (*d*), Article 25, that a negative charge is equivalent to a deficit of positive electricity and a positive charge is equivalent to a deficit of negative electricity.)

Hence, as far as the magnetic effects of an electric current are concerned, it is immaterial whether an electric current be looked upon as a stream of positive electricity moving in one direction, or a stream of negative electricity moving in the opposite direction, or a combination of two such streams.   For example, if positive electricity moves through a wire from $A$ to $B$ at the rate of 5 units per second, and negative electricity moves through this wire from $B$ to $A$ at the rate of 3 units per second, the magnetic effects produced are exactly the same as would be produced by $5 + 3 = 8$ units of positive electricity per second moving from $A$ to $B$, or to $5 + 3 = 8$ units of negative electricity per second moving in the opposite direction, that is, from $B$ to $A$.

Therefore, in the above definition of the intensity of an electric current, by the quantity of electricity which flows through a

given cross-section per unit of time is to be understood the sum of the positive electricity which moves through this area in one direction plus arithmetically the negative electricity which moves through this area in the opposite direction.

The definition of the intensity of an electric current may then be expressed mathematically as follows: Let

$q$ = the quantity of positive electricity which moves through a given area $S$ in time $t$, plus the quantity of negative electricity which moves through this area in the opposite direction in this same interval. In order to avoid circumlocution, the quantity $q$ as thus defined will hereafter be referred to simply as the total quantity of electricity transferred across the area $S$ in time $t$,

$I$ = the current through the area $S$ during this interval $t$. Then, when the rate of flow is uniform,

$$I = \frac{q}{t} \tag{1}$$

When the rate of flow is not uniform, this relation gives the *average* value of the current during the interval $t$. The value of the current *at any instant* during this interval is defined by the relation

$$i = \frac{dq}{dt} \tag{1a}$$

where $dq$ represents the total quantity of electricity transferred across the given area $S$ in an infinitesimal interval of time $dt$, measured from the instant under consideration.

The *direction* of the electric current at any point in its path is taken as the direction in which the stream of positive electricity moves past this point, or, what amounts to the same thing, as the direction opposite to that in which the stream of negative electricity moves past the point. This definition of the direction of an electric current is fundamental, but can be applied practically only in certain very special cases. However, experiment shows that the direction of the current in a wire may always be determined by the following very simple test:

Place under the wire a small magnetic needle (*e.g.*, an ordinary pocket compass). The needle will take up a position at right angles to the wire as shown in Fig. 13. Looking across the wire in the direction which the needle points (*i.e.*, in the direction

from its normally south-seeking to its normally north-seeking end), the current in the wire is from left to right.

When the intensity of an electric current remains constant with respect to time for an appreciable interval, the current is said to be "continuous" during that interval. When the intensity varies with time, the current is said to be a variable current. It is usual to represent a varying current by the symbol $i$ and a continuous current by the symbol $I$.

The following definitions are quoted from the Standardization Rules of the American Institute of Electrical Engineers:

**Direct Current.**—A unidirectional current. As ordinarily used, the term designates a practically non-pulsating current.

**Pulsating Current.**—A current which pulsates regularly in magnitude. As ordinarily employed, the term refers to unidirectional current.

**Continuous Current.**—A practically non-pulsating direct current.
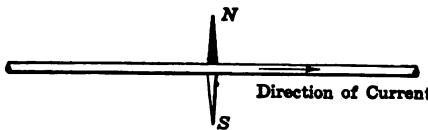


FIG. 13.

**Alternating Current.**—A current which alternates regularly in direction. Unless distinctly otherwise specified, the term "alternating current" refers to a periodic current with successive half waves of the same shape and area.

**Oscillating Current.**—A periodic current whose frequency is determined by the constants of the circuit or circuits.

The term "continuous current" is seldom used in this country. Instead a non-varying (or practically non-varying) current is called a "direct current." The term "direct current" will be so used throughout this book. A varying unidirectional current will be called a "unidirectional" current and not a "direct current."

The practical unit of electric-current intensity is called the "ampere," and is equal to a flow of 1 coulomb of electricity per second (see also Article 32).

**Problem 2.**—Referring to Problem 1, if the time required for the flow of electricity between the respective conductors is 0.01 second, what will be the direction and average intensity of the currents, in amperes, through each

of the three batteries during this interval, assuming all three switches to be closed simultaneously?

*Answer.*—A current of 10 amperes from *A* to *B*, a current of 7.5 amperes from *B* to *C*, and a current of 6 amperes from *C* to *A*.

**29. Kirchhoff's First Law for Conducting Networks.**—When the total quantity of electricity which flows up to a surface in any given interval of time is equal to the total quantity of electricity which flows away from that surface in the same interval, no electricity accumulates at that surface. Hence, from the relation between quantity of electricity and current intensity stated in the preceding article, it follows that when the intensity of the current in the substance on one side of a surface is equal to the intensity of the current in the substance on the other side of this surface, there is no change in the total quantity of electricity, or electric charge, at this surface. Conversely, when there is no change in the electric charge on a given surface, the intensity of the current is the same on the two sides of this surface.

Experiment shows that in the case of any network of conductors carrying *direct* currents (using the term direct current, as explained above, to mean a current whose intensity does not vary with time), no change takes place in the total quantity of electricity at any part of the network. That is, as much electricity flows into each part of any conductor as flows out of it. Hence, for direct currents, the total current coming up to any junction point or surface is always equal to the total current leaving that junction point or surface.

Or, considering a current which *leaves* a point or surface, as a *negative* current *entering* that point or surface, an equivalent statement of this relation is that the algebraic sum of the currents entering any point or surface is always zero, provided these currents are all direct currents.

This fundamental relation, which is known as **Kirchhoff's First Law for Conducting Networks,** may be stated mathematically as follows: Let $I_1, I_2, I_3 \ldots I_n$ be the currents in any number ($n$) of conductors which are connected at a common junction point or surface, and let the currents which *enter* this point or surface be represented by positive numbers, and let the currents which *leave* this point or surface be represented by negative numbers. An $I$ may then stand for either a positive or a negative number, depending upon whether the current which it represents enters or leaves the point. Then

$$I_1 + I_2 + I_3 + \ldots + I_n = 0 \tag{2}$$

or

$$\sum_1^n I = 0 \tag{2a}$$

When the currents vary with time, experiment shows that a positive or a negative charge may accumulate at the surface of contact between dissimilar substances, particularly at the surface of contact between a conductor and a good insulator or dielectric. This means that while the currents are varying, the rate at which electricity flows up to such a surface on one side may be different from that at which it flows away from the other side of this surface.

For example, let $i$ be the current (rate of flow of electricity) entering one side of such a surface, and $i'$ be the current leaving the other side of this surface. Then from equation (1a)

$$i - i' = \frac{dq}{dt} \tag{3}$$

and the total charge which accumulates on this surface in an interval of time $t$ is

$$q = \int_0^t (i - i')\, dt \tag{3a}$$

where the symbol $\int_0^t$ indicates the summation of the products $(i - i')dt$ for the successive infinitesimal intervals which make up the interval $t$. In this equation a positive value of the integral signifies a positive charge, and a negative value of the integral a negative charge.

Hence, whenever the charge on a surface is changing, the current coming up to that surface is not equal to the current which leaves that surface. In the special case of the surface of contact between a conductor and an insulator or dielectric in which there is no appreciable current, equation (3a) becomes

$$q = \int_0^t i\, dt \tag{3b}$$

where $i$ is the current coming up to this surface through the conductor. When there is an appreciable current through the insulator, equation (3a) must be used. The flow of electricity through a dielectric is frequently referred to as the "leakage" current through this dielectric.

4

From these relations it follows that Kirchhoff's First Law is not only applicable to direct currents but also to any junction point or surface in a network carrying varying currents, *provided there is no change in the charge at this junction point or surface* (see also Chapter XIV).

**Problem 3.**—A certain conducting network has four junction points, namely, *A*, *B*, *C* and *D*, each of which is connected to the other three. The current from *C* to *A* is 15 amperes, the current *A* to *B* is 10 amperes, and the current from *D* to *B* is 8 amperes.

(*a*) What are the values and directions of the currents in the other branches of the network, if all currents are non-varying with respect to time?

(*b*) Draw a circuit diagram and indicate on it the value and direction of each current.

*Answer.*—(*a*) 5 amperes from *A* to *D*, 18 amperes from *B* to *C* and 3 amperes from *C* to *D*.

**Problem 4.**—Two metal plates, separated by a thin sheet of paper, are connected by insulated wires to the terminals of a dry cell. At the instant this connection is made a current flows through the wires and cell, rising to a maximum value and then dropping to zero. The direction of the current *through the cell* is from its negative to its positive terminal. The total time of this flow is 0.002 second, and during this interval the average value of the current in each wire is 0.03 ampere. No electricity flows through the sheet of paper separating the plates.

(*a*) Draw a diagram of the circuit, showing the plates, paper, connecting wires and cell, and indicate the amount and sign of the charge on each plate when the flow of electricity in the circuit has ceased. (*b*) What will happen to the charge on each plate if the wires are disconnected from them?

*Answer.*—(*a*) The plate connected to the negative terminal of the cell will become negatively charged with 0.00006 coulombs, and the plate connected to the positive terminal of the cell will become positively charged with 0.00006 coulombs. (*b*) The charges will remain on the plates when the wires are disconnected. (NOTE.—The cell may be looked upon merely as producing a force which causes a transfer of 0.00006 coulombs of positive electricity from the metal plate which is connected to the negative terminal, to the plate which is connected to the positive terminal; or, as producing a transfer of an equal quantity of negative electricity in the opposite direction.)

**30. Series and Parallel Connections.**—When two or more insulated conductors are connected end to end, they are said to be in series. When two or more conductors are in contact at two common junction points, but are insulated from one another between these points, they are said to be in parallel.

Fig. 14*A* shows three conductors in series, and Fig. 14*B*, three conductors in parallel. Several conductors in parallel may be connected in series with one or more conductors, such as 4 and

5 in Fig. 14*B*.   The several conductors are then said to be con-
nected in series-parallel.

The most general type of circuit for a continuous flow of elec-
tricity (direct currents) is a network of conductors, each mesh
of which is a closed conducting loop, such as is shown diagram-
matically in Fig. 22.

From Kirchhoff's First Law, the currents in each of several
conductors in series are all equal, provided there is no change in
the charges on these conductors.   For example, in Fig. 14*A*,

$$i_1 = i_2 = i_3$$

Kirchhoff's First Law applied to Fig. 14*B* gives

$$i_4 = i_1 + i_2 + i_3 = i_5$$

provided there is no change in the charges on these conductors.



Series Connection         Parallel Connection
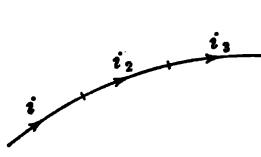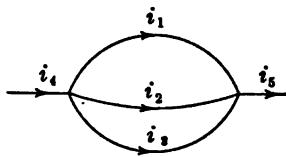
FIG. 14*A*.                 FIG. 14*B*.

Note that the arrows in a circuit diagram, such as Figs. 14*A*
and 14*B*, are used to indicate the positive *senses* of the currents
in the several parts of the circuit.   In any problem dealing with
an electric network it is always well to designate the positive
sense of each current in some such way.   When the actual direc-
tion of the current is not known, either sense may be assumed as
the positive sense.   In the final answer to the problem a negative
value of any current is then to be interrupted as a current which
flows in the direction opposite to that assumed as its positive
sense.

**Problem 5.**—Between two junction points *A* and *B* in an electric circuit
there are four parallel branches.   The total current coming up to the junc-
tion point *A* is 100 amperes, and the currents leaving *A* in three of the
branches are respectively 25, 35 and 20 amperes.

(a) What is the current in the fourth branch between *A* and *B*?   (b)
What is the total current leaving *B*?

*Answer.*—(a) 20 amperes.   (b) 100 amperes.

**31. Systems of Electric Units.**—Neither an electric charge nor an electric current can be measured directly, but only in terms of some measurable effect which either produces. From the fundamental definition of the intensity of an electric current through any cross-section of its circuit as the total quantity of electricity which flows through this cross-section *per unit of time*, it follows that when a unit of electric charge is definitely specified in terms of some measurable effect produced by a charge, then a corresponding unit of current intensity may be derived from this unit of charge. Conversely, when a unit of current intensity is definitely specified in terms of some measurable effect produced by a current, a corresponding unit of charge may be derived therefrom.

Unfortunately three different effects have been chosen as the basis of as many systems of electrical units. These three effects are (1) the mutual force of attraction or repulsion between charged bodies, (2) the mutual force between a conductor carrying an electric current and the end, or "pole," of a magnetic needle in its vicinity, and (3) the mass of silver deposited · per second when a current is passed through a solution of nitrate of silver contained in a silver "voltameter."

The system of units based on the first effect and on the centimeter, gram and second as the units of space, mass and time respectively, is called the c.g.s. *electrostatic* system of units. The system of units based on the second effect and on the centimeter, gram and second, is called the c.g.s. *electromagnetic* system of units. Either of these two systems of units is sometimes referred to as an "absolute" system of units, but the term "absolute unit" is usually reserved for the units in the c.g.s. electromagnetic system.

The third of the effects just listed is taken as the basis of the so-called practical system of electrical units. As already noted, the unit of current intensity in the practical system of units is called the ampere, and most of the units derived therefrom have been given specific names, such as "coulomb" for the practical unit of charge or quantity of electricity, "volt" for the unit of electric "potential," etc. In order to avoid circumlocution, the corresponding units in the c.g.s. electrostatic system may be designated by these same names with the prefix "stat," and the corresponding units in the c.g.s. electromagnetic system by the

prefix "ab" (standing for "absolute"). This usage of these prefixes, though not universal, is coming more and more into favor, and will be adopted throughout this book.

Of these three systems of units the one in which the units can be most conveniently specified is the practical system. Historically, the other two systems of units were developed first, but the exact specification of the fundamental unit in either of these systems in such a manner that this unit may be physically produced in the laboratory is not a simple matter.

**32. The International Ampere and the International Coulomb.** —As explained in Article 27, when an electric current is established through an acid or salt solution, chemical changes usually take place at the surface of contact between the solution and the conductor through which the current is led into and out of the solution. In particular, when a current is established through a solution of silver nitrate in water, and passes out of the solution through a plate of silver (or through a piece of silver of any form), experiment shows that silver is deposited from the solution onto the silver plate. The solution, rods and the containing vessel is called a "voltameter."

Experiment shows that the mass of silver which is thus deposited by an unvarying current is directly proportional to the time during which this current flows, and that the *rate* at which silver is deposited from such a solution may be taken as the measure of the *rate* at which electricity is conducted through the solution. Hence, the number of grams of silver deposited per second on the silver plate may be taken as a measure of the intensity of this current.

The exact specification of the practical unit of current intensity, the ampere, has been a subject of some controversy, but at the international electric congress held in London in 1908, the following definition was adopted:

"The international ampere[1] is the unvarying electric current which,

[1]The legal definition of the international ampere in the United States (Act of Congress passed July 12, 1894) is as follows:

"The unit of current shall be what is known as the international ampere, which is one-tenth of the unit of current of the centimeter-gram-second system of electromagnetic units, and is the practical equivalent of the unvarying current, which, when passed through a solution of nitrate of silver in water in accordance with standard specifications, deposits silver at the rate

when passed through a solution of nitrate of silver in water, deposits silver at the rate of 0.00111800 of a gram per second."

For information in regard to the specifications for the silver voltameter, see *Circular* No. 60 of the U. S. Bureau of Standards.

The measurement of electric current by determining the amount of silver deposited is too cumbersome a process for practical use. Instead, measuring devices based on the magnetic action of an electric current (*e.g.*, galvanometers and ammeters; see Article 33) are usually employed. Such instruments, however, must always be calibrated by comparison with certain secondary standards, which in turn have been calibrated, either directly or indirectly, in terms of some primary standard, such as the silver voltameter, or standard ohm and standard cell (see Articles 36 and 73).

To enable the reader to form a rough conception of the magnitude of an ampere, the following figures are given. An ordinary carbon-filament, 16-candlepower, incandescent lamp takes a current of about 0.5 ampere. The current taken by an ordinary city trolley car when running at full speed on a level track is about 100 amperes. The current taken by a single New York Central electric locomotive when starting a train is about 3000 amperes. The "voice" current in an ordinary telephone receiver is about 0.001 ampere.

Very small currents are usually expressed in milliamperes (1 milliampere equals 0.001 ampere) or in microamperes (1 microampere equals 0.000001 ampere). The "voice" current in a telephone receiver is then about 1 milliampere.

From the fundamental relation between the total quantity of electricity conducted through a substance and the intensity of

---

of one thousand one hundred and eighteen millionths (0.001118) of a gram per second."

The difference between the U. S. legal definition and that of the London Conference is that the legal definition states that the international ampere is *practically equivalent to*, whereas the London Conference defines the international ampere as *exactly equal to*, the current which deposits silver at the rate of 0.001118 gram per second. The legal definition defines the international ampere specifically in terms of c.g.s. electromagnetic unit, whereas the London Conference leaves to experiment the determination of the exact relation between the two units. This experimental relation is, to within one-twentieth of 1 per cent., that 1 ampere = 0.1 c.g.s. electromagnetic unit.

the current in this substance (Article 7), the practical unit of quantity of electricity is the quantity of electricity conducted through a surface by 1 ampere flowing for 1 second. As already stated, this unit is called the coulomb. A more commonly used unit of quantity of electricity is the ampere-hour, which is the quantity of electricity transferred by 1 ampere in 1 hour. For example, a continuous current of 100 amperes flowing for 5 hours transfers 500 ampere-hours of electricity through each section of the circuit. One ampere-hour is equal to 3600 coulombs.

**Problem 6.**—A current of 75 amperes flowing through a silver voltameter for a certain length of time deposits 100 grams of silver on the cathode.

(a) For how long a time does this current flow? (b) How many ampere-hours of electricity pass through the voltameter in this interval? (c) How much silver does each ampere-hour deposit?

*Answer.*—(a) 19.9 minutes. (b) 24.9 ampere-hours. (c) 4.025 grams.

**33. Galvanometers, Ammeters and Electrodynamometers.**— The instruments commonly employed in practice for the determination of the intensities of direct currents are of three fundamental types, which may be designated as the moving-magnet type, the moving-coil type, and the two-coil type.

The indications of these instruments, unless they are constructed with great precision and in such a manner that the forces acting on the moving member may be accurately calculated, cannot be evaluated in amperes without comparing them with a silver voltameter or other primary standard, or with some other instrument whose "constants" are accurately known. The determination of the true values corresponding to the indications of an instrument is commonly spoken of as "calibrating" the instrument.

Current-measuring instruments may also be classified as galvanometers or ammeters. The distinction usualy made between a galvanometer and an ammeter is that the indication of a galvanometer is determined by noting the motion over a fixed scale of a spot of light reflected from a small mirror fixed to the moving element, or by noting through a telescope the motion of the reflection from this mirror of a fixed scale, whereas the indication of an ammeter is read by noting on a fixed scale attached to the case of the instrument the position of a pointer which is attached to the moving element.

**The galvanometer form of the moving-magnet type** of instru-

ment consists essentially of a permanently magnetized needle suspended inside a coil of insulated wire, which may be connected in series with the conductor in which the current to be measured is established. The current in the coil exerts a mechanical couple on the needle (Article 129), which deflects this needle from its normal position of equilibrium, which latter may be due either to the earth's magnetic field, to a control magnet, or to the torsion in the fiber from which the needle is suspended.

Although the needle galvanometer may be made extremely sensitive, it has the serious defect of not holding its calibration
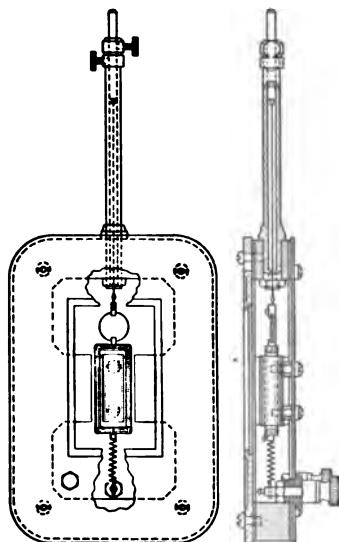


FIG. 15.—D'Arsonval galvanometer.

for any great length of time. This is due to the effects of temperature, moisture, etc., on the fiber supporting the magnetic needle, and also to external magnetic disturbances. When suitably mounted and magnetically screened, however, it may be used for the most delicate measurements.

The moving-coil, or D'Arsonval, type of galvanometer (Fig. 15) is much superior to the moving-needle type, except where great sensitiveness is required. It consists of a coil of fine insulated wire suspended by a metal fiber or thin metal strip, between the two poles of a powerful permanent magnet made in the form of a horseshoe. The fixed end of the metal strip

supporting the coil is connected to a suitable terminal, or binding-post, on the frame of the instrument, and the other end to one end of the wire forming the coil. The other end of the coil is connected through a second metal strip, usually wound in the form of a light spiral spring, to a second insulated terminal or binding-post. When the two terminals of such an instrument are connected in series with the circuit in which the current to be measured is established, the same current is also established in the wire forming the moving coil, and the mechanical couple produced on this coil by the permanent magnet causes a deflection of the coil, approximately proportional to the intensity of the current (see Article 132).
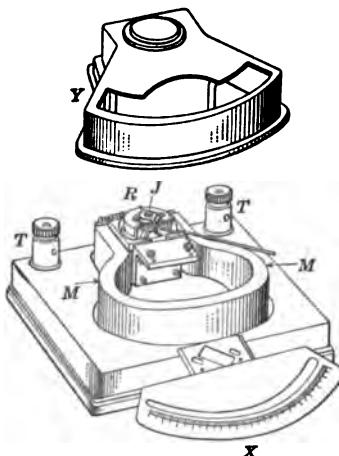


Fig. 16.—Weston ammeter.

**The ammeter type of moving-magnet instrument** usually consists of a soft-iron core or vane which is so mounted that it is free to move inside a coil of wire. The current to be measured is established through this coil, and under the influence of the current the core or vane becomes magnetized, and is acted upon by a deflecting force due to the current. This type of ammeter can also be used for measuring alternating currents, and, in fact, is much more reliable for alternating than for direct currents.

**The ammeter type of moving-coil instrument** (Fig. 16) is similar to the D'Arsonval galvanometer, the only important distinction being that the moving coil is provided with a fine pivot which rests in a jewel bearing, the current being led into

and out of the moving coil by means of two small flat spiral springs connected to the two ends of the coil respectively and also to the terminals on the outside of the case of the instrument. These springs also serve to hold the coil normally in a definite position. Moving-coil direct-current ammeters are not suitable for alternating-current measurements (see also Article 44).

Fig. 16 shows the construction of the particular form of this type of ammeter made by the Weston Electrical Instrument Co. In the figure, the instrument proper, the case $Y$ and the scale $X$ are shown separate, in order to show the working parts of the device. When actually used the scale $X$ is permanently attached to the magnet $M$ under the pointer, and the case $Y$ covers the whole.



Mercury Cups
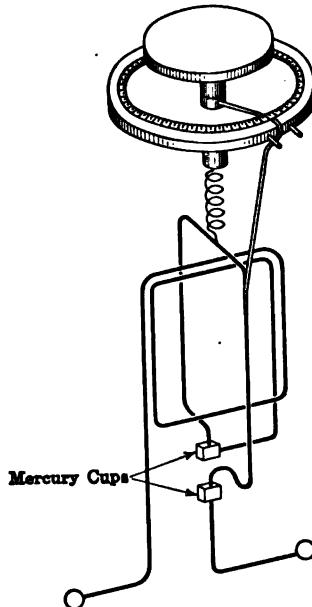
FIG. 17.—Electrodynamometer.

**The two-coil type** of current-measuring instrument is made in two forms, known as the electrodynamometer and the current balance respectively. In both types the current to be measured passes through two or more coils in series, one coil or set of coils being fixed, and the other coil, or set of coils, being free to move under the action of the force which is exerted by one set of coils on the other when a current flows through them.

In the **electrodynamometer** (Fig. 17) the moving coil is suspended or pivoted so that it can turn about a vertical axis, and the controlling force which opposes the force of the current is due either to the torsion of the suspension fiber or to a spiral spring such as is used in a Weston ammeter. In their normal position the planes of the moving and fixed coils are at right angles. The moving element of an electrodynamometer may be provided either with a mirror, or with a pointer which moves over a graduated scale, or a torsion head may be used to turn the needle back to its "zero" position, in which case the angle through which this head is turned is read on the scale.

**The current balance** (Fig. 18) is similar in principle to the electrodynamometer, except that two sets of fixed horizontal



Electrical Connections

Movable Coil
Fixed Coils

Mechanical Arrangement
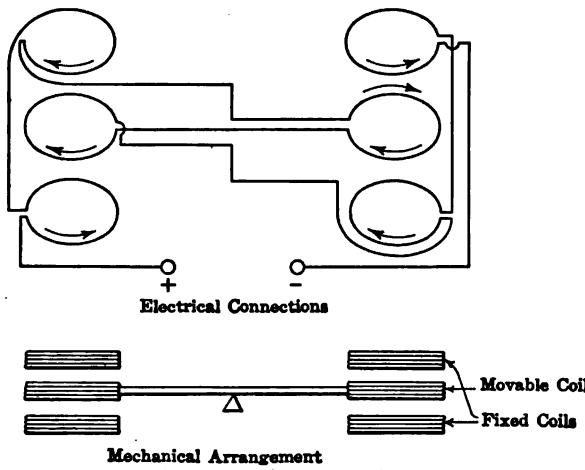
FIG. 18.—Current balance.

coils are used, with a set of movable coils suspended between the sets of fixed coils, and the controlling force is gravity. The arrangement of a current balance is similar to that of an ordinary double-platform balance, the mutual force between the two sets of coils being "weighed" in exactly the same manner that a mass is weighed (see also Article 131).

In a carefully designed and constructed current balance, the force between the two sets of coils corresponding to any current through these coils may be accurately calculated in terms of the intensity of this current in c.g.s. electromagnetic units. A current balance so designed is called an "absolute" current

balance, and by the use of such an instrument the relation be-
tween the international ampere (as defined in Article 32) and the
c.g.s. electromagnetic unit may be experimentally determined.

**34. Electric Energy and Electric Power.**—The student is ad-
vised to re-read carefully, at this point, Articles 15, 16, and 19
of Chapter I.

As illustrated in Article 24, when an electric current is estab-
lished in a wire connected to the terminals of a dry cell, heat
energy is developed in the wire and also in the conductors forming
the battery, and the materials of the cell lose chemical energy.
From the principle of the conservation of energy, it is perfectly
rational (a) to look upon the chemical energy which is lost by the
battery as being converted within the battery into another form
of energy, which may be called " electric energy," (b) to look upon
the current as being the means whereby this electric energy is
transferred from one part of the circuit to another, and (c) to
look upon the heat energy which is developed in any part of the
circuit as being due to the conversion, within this part of the
circuit, of part of this electric energy into heat energy.

The production, transfer and conversion of electric energy in
the simple circuit just described is typical of the production,
transfer and conversion of electric energy in every electric circuit,
except that other forms of energy than chemical energy may
be converted into electric energy, and electric energy may be
converted into other forms than heat energy.

It should be carefully noted, however, that although the flow
of electricity is always accompanied by a transfer of energy,
electricity is not energy, just as matter is not energy. Experi-
ment justifies the conclusion that the energy of a given quantity,
or " portion," of electricity depends upon its position and motion
relative to other portions of electricity, just as the energy of a
portion of matter depends upon its position and motion relative
to other portions of matter.

As noted in Article 19, the *rate* at which any kind of energy is
developed, transferred or converted into some other form is
called the "power" developed, transferred or converted. Hence,
the rate at which electric energy is developed in any part of an
electric circuit is called the electric power developed, or generated,
in this part of the circuit. Similarly, the rate at which electric
energy is transferred by an electric current *from* any device is

called the electric power output of this device, and the rate at which electric energy is transferred *to* any device is called the electric power input to this device.

**Problem 7.**—When a certain coil of insulated wire is connected to the two terminals of a dry cell, a steady current of 3 amperes is established in this coil. The electric energy developed within the cell by the chemical changes which take place is 1.5 joules (1 joule = $10^7$ ergs = 0.7376 foot-pound) for each coulomb of electricity which flows through it. Of this energy developed within the cell 10 per cent. is dissipated as heat energy within the cell itself.

(*a*) How much electric energy is developed within the cell in 5 minutes? (*b*) How much electric energy is transferred from the cell to the coil of wire in this interval? (*c*) How much electric power is developed within the cell? Express this both in joules per second and in horsepower. (*d*) How much electric power is converted into heat within the cell itself? (*e*) How much electric energy is transferred to the coil of wire per coulomb of electricity which passes through it?

*Answer.*—(*a*) 1350 joules. (*b*) 1215 joules. (*c*) 4.5 joules per second = 0.00603 horsepower. (*d*) 0.45 joules per second. (*e*) 1.35 joules per coulomb.

**Problem 8.**—The electric power supplied to an alternating-current motor varies with time according to the law

$$p = P_o\left[\cos\theta - \cos\left(\frac{4\pi t}{T} + \theta\right)\right]$$

where $P_o$ and $\theta$ are constants ($\theta$ being less than $\frac{\pi}{2}$), $T$ is the time required for the current to pass through a complete cycle of values, *i.e.*, the "period" of the current, and $t$ is time measured from the instant at which the current is zero.

(*a*) Plot this function with time as abscissas and instantaneous power as ordinates, using for time such a scale that the period $T$ equals 360. (*b*) What is the average electric power input to the motor during an interval of time equal to $T$? (*c*) What is the maximum power input to the motor, and for what values of $t$ does this maximum input occur? (*d*) Does the motor give out electric energy during any portion of the cycle of the current, and if so, between what values of $t$?

*Answer.*—(*b*) $P_o \cos\theta$. (*c*) $P_o (1 + \cos\theta)$ for $t = \left(\frac{1}{4} - \frac{\theta}{4\pi}\right) T$ and for $t = \left(\frac{3}{4} - \frac{\theta}{4\pi}\right) T$ and for each half period thereafter. (*d*) Yes, for all values of $t$ between $\left(\frac{1}{2} - \frac{\theta}{2\pi}\right) T$ and $\frac{1}{2} T$ and for all values of $t$ between $\left(1 - \frac{\theta}{2\pi}\right) T$ and $T$.

**35. Electric Resistance.—Joule's Law.**—As already noted, the flow of electricity through a conductor is always accompanied by a dissipation of electric energy as heat energy within the conductor. This phenomenon may be looked upon as due to a

frictional resistance offered by the conductor to the flow of electricity through it. Compare with the flow of water through a pipe or other channel; see Article 21.

Experiment shows that in the case of a wire of uniform material, kept at a constant uniform temperature, the rate at which heat energy is dissipated in a given length, between any two points $A$ and $B$ say, when an electric current is established in it, is proportional to the square of the strength of this current.

That is, calling $p_h$ the rate at which heat energy is developed in the wire between the points $A$ and $B$, at any given instant, and $i$ the intensity of the current in the wire at this instant, then

$$p_h = ri^2 \tag{4}$$

where $r$ is a factor whose value depends upon the dimensions and temperature of the wire, the nature of the substance of the wire, and the units in which $p_h$ and $i$ are measured, but is *independent* of the *strength* of the current. This factor $r$, however, does depend upon the **rate of variation** of the current, particularly if the conductor has a large cross-section.

The factor $r$ defined by equation (4), namely, **the quantity by which the square of the current in a given conductor must be multiplied in order to give the rate at which heat energy is developed in it, is called the "electric resistance"** of this conductor.

The resistance of a given conductor depends upon the surfaces or points at which the current enters and leaves it (see Article 54). By the resistance of a wire is meant its resistance to a current which enters at one end and leaves at the other.

The resistance of a conductor to a varying or alternating current is greater than its resistance to a direct (non-varying) current, although in many cases this difference is inappreciable (see Chapter XV). The resistance of a conductor to a direct-current is often called its "ohmic resistance," to distinguish this resistance from its resistance to a varying current.

The experimental fact represented by equation (4) is known as Joule's Law, from the name of the scientist who first clearly enunciated this relation.

**36. Units of Electric Resistance, Power and Energy.**—The practical unit of electric resistance now almost universally used is the "international ohm" defined by the London Congress of 1908 as follows:

"The international ohm[1] is the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14.4521 grams in mass, of a constant cross-sectional area and of a length of 106.300 centimeters."

The column of mercury thus specified is approximately 1 square millimeter in cross-section. The choice of the length of 106.300 centimeters arises from the fact that in a column of mercury of this length and weight and at 0°C. the rate at which 1 ampere develops heat energy is practically 1 joule per second, or 1 watt.

The mercury column is used only as a primary standard. As secondary, or ordinary laboratory standards of resistance, wire coils are employed, so designed that their resistances are exact decimal multiples or submultiples of an ohm, *i.e.*, 1, 10, 100, etc., or 0.1, 0.01 etc., ohm. A set of such coils mounted in a box and connected in series between two or more binding-posts is called a "resistance box." Suitable plugs or switches are provided on such boxes so that any desired resistance may be inserted between the binding-posts.

To express small resistances a unit one-millionth of the size of an ohm is ordinarily used; this unit is called the "microhm." To express large resistances a unit one million times the size of an ohm is frequently used; this unit is called the "megohm." Hence:

$$1 \text{ ohm} = 10^6 \text{ microhms.}$$
$$1 \text{ megohm} = 10^6 \text{ ohms.}$$

The c.g.s. electromagnetic unit of resistance may be defined as that resistance in which 1 abampere ( = 10 amperes) develops heat at the rate of 1 erg per second. This unit of resistance is

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[1]This definition of the international ohm is in complete agreement with the definition legalized by the Act of Congress approved July 12, 1894, the wording of which is:

"The unit of resistance shall be what is known as the international ohm, which is substantially equal to one thousand million units of resistance of the centimeter-gram-second system of electromagnetic units, and is represented by the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, fourteen and four thousand five hundred and twenty-one ten-thousandths grams in mass, of a constant cross-sectional area, and of the length of one hundred and six and three-tenths centimeters."

called the abohm.   The relation between the ohm and the abohm is that

$$1 \text{ ohm} = 10^9 \text{ abohms}.$$

As noted in Article 19, the unit of power corresponding to 1 joule per second is called the "watt."   The watt as thus defined is called the "absolute" watt as distinguished from the "international" watt, which is the power developed as heat in a resistance of 1 international ohm by a direct current of 1 international ampere (see also Article 39).   For all practical purposes the absolute watt and international watt may be considered as identical, the exact relation being that 1 absolute watt = 0.99966 international watt.

Electric power is usually expressed in kilowatts, abbreviated kw.   A kilowatt, by definition, is 1000 watts.   A kilowatt is approximately equal to 1.34 horsepower.   The exact relation (to four significant figures) between the horsepower and the watt is that 1 standard[1] horsepower is equal to 745.7 watts.   An approximate relation which is easily remembered is that 1 kilowatt equals 4/3 horsepower.   Large amounts of power are frequently expressed in "megawatts," a megawatt being by definition one million watts.   To summarize:

1 horsepower = 745.7 watts.
1 kilowatt    = 1000 watts.
1 megawatt  = 1000 kilowatts.

Electric energy is usually expressed in watt-seconds, in watt-hours, or in kilowatt-hours.   By a watt-second, which is equal to 1 joule, is meant the energy corresponding to 1 watt for 1 second.   By a watt-hour is meant the energy corresponding to 1 watt for 1 hour.   By a kilowatt-hour is meant the energy corresponding to 1 kilowatt for 1 hour.   The kilowatt-hour is the unit of energy usually employed in practical work.   In England the name "kelvin" is frequently used to designate a kilowatt-hour.

Problem 9.—In a private residence there are 50 incandescent electric lamps, each of which takes 40 watts of electric power when lighted.   During each day of the month of January, five of these lamps are lighted from 5 A.M. to 8 A.M., seven from 4 P.M. to 6 P.M., ten from 6 P.M. to 11 P.M., and two from

[1] By a standard horsepower is meant 550 foot-pounds per second at a point where the acceleration due to gravity is 980.665 centimeters per second per second.

11 P.M. to 1 A.M. If the owner of the house pays 10 cents a kilowatt-hour for electric energy, what will be his bill for lighting for this month?

*Answer.*—$10.29.

**Problem 10.**—An electric stove when connected to the electric circuit in a house takes a current of 25 amperes, and the electric power absorbed by the stove is 2.8 kilowatts.

(a) What is the electric resistance of this stove? (b) If 30 per cent. of the heat developed by this stove is absorbed by a gallon of water in a kettle placed on it, how long will it take this water to boil, the temperature of the water at the start being 60°F.? (Neglect the loss of heat due to radiation from the kettle and the specific heat of the kettle.[1])

*Answer.*—(a) 4.48 ohms. (b) 26.6 minutes.

**Problem 11.**—An electric motor develops 50 horsepower at its pulley. The efficiency of the motor (*i.e.*, the ratio of its mechanical power output to its electric power input, when both are expressed in the same units) is 85 per cent. How many kilowatt-hours of electric energy must be supplied to the motor to develop 50 horsepower at its pulley for 10 hours?

*Answer.*—439 kilowatt-hours.

**37. Sources and Receivers of Electric Energy.**—As noted in Article 34, the flow of electricity in an electric circuit is always accompanied by a transfer of electric energy from one part of the circuit to another. Any part of an electric circuit from which electric energy is transferred by the current is called a "source" of electric energy, and any part of an electric circuit to which electric energy is transferred by the current is called a "receiver" of electric energy. Experiment shows that certain devices, such as an electric dynamo and a storage cell, may act either as a source or as a receiver of electric energy, depending upon the direction in which the current flows through them.

The wires which connect a source and a receiver of electric energy, and which serve as the path over which the electricity flows around the circuit from one to the other, themselves absorb a certain (but in general relatively small) amount of electric energy, which is dissipated as heat energy. Strictly speaking, these wires are then receivers of electric energy. However, the wires between a source and a receiver are not classed as a receiver, but are commonly referred to simply as the connecting wires, or, when the source and the receiver are at a considerable distance apart, as the "mains," or the "transmission line" between the two.

[1] For the necessary constants and conversion factors to solve numerical problems of this kind see any engineers' handbook, or any book of physical constants and conversion factors.

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As a relatively simple example of a device which may serve either as a source or a receiver of electric energy may be taken the lead storage cell. Such a cell consists of two sets of lead plates immersed in a solution of sulphuric acid in water. The plates of each set are connected in parallel, and each set of plates may therefore be considered as a single plate. The body of each plate is metallic lead, but the surface of one, called the "positive" plate, is covered with lead peroxide, giving it a reddish-brown surface, and the surface of the other, called the "negative" plate, is spongy lead of a grayish color.

When the terminals of such a cell are connected by an external conductor, as shown in Fig. 19, a current will flow through the closed circuit thus formed, flowing out of the battery at its positive plate and into the battery at its negative plate. This flow of electricity is accompanied by a chemical reaction between the lead peroxide and the sulphuric acid and between the lead and
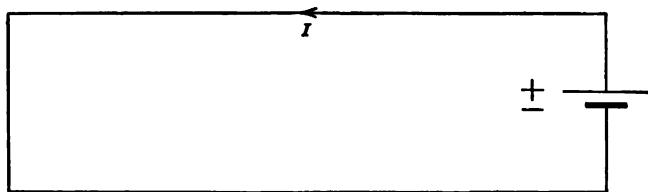


FIG. 19.

sulphuric acid, resulting in the formation of lead sulphate on each plate and in the formation of water in the electrolyte, which latter consequently decreases in specific gravity. The final result of this chemical reaction may be represented by the chemical equation

$$PbO_2 + 2H_2SO_4 + Pb = 2PbSO_4 + 2H_2O.$$

Experiment shows that such a chemical reaction without a flow of electricity would be accompanied by the development of a definite amount of heat energy, from which fact is drawn the conclusion that a given total weight of lead peroxide, sulphuric acid and lead has more "chemical" energy than the corresponding total weight of lead sulphate and water. However, experiment also shows that when a flow of electricity is produced by such a chemical reaction, there is only a relatively small change in the temperature of the cell, showing that practically all the

energy which would otherwise appear as heat is converted within the cell into some other form. This other form of energy, as noted above, is conveniently designated as electric energy.

These experimental facts may then be briefly described by the statement that within the cell chemical energy is converted into electric energy. The major portion of the electric energy developed within the cell is transferred by the current to the external conductor which connects its terminals, and is there dissipated as heat or converted into some other form of energy. A small portion of the electric energy developed within the cell is also dissipated as heat energy within the cell itself, due to the resistance of the conductors of which it consists.

As long as the cell continues to give out electric energy it is said to be "discharging." The reader should bear in mind, however, that the cell does not discharge electricity, but *electric*
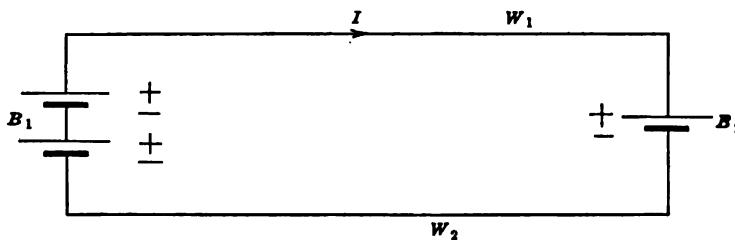


FIG. 20.

*energy.* As much electricity flows into the cell at its negative terminal as flows out at its positive terminal. However, it is common practice to speak of a cell as discharging so many amperes or so many ampere-hours, meaning thereby the specified number of amperes, or ampere-hours, flows through the cell from its negative to its positive plate.

When a storage cell is partially or completely discharged, *i.e.*, when a part or all of the lead peroxide has been converted into lead sulphate, it may be recharged by suitably connecting it to some other source of electric energy. For example, a partially discharged cell $B_2$ (see Fig. 20) may be recharged by connecting it, through the wires $W_1$ and $W_2$, to a battery $B_1$ consisting of two fully charged cells connected in series. In the battery $B_1$ one cell has its positive terminal connected to the negative terminal of the other cell. The positive terminal of the partially discharged cell $B_2$ is connected to the free positive terminal of

the battery $B_1$, and the negative terminal of $B_2$ is connected to the free negative terminal of the battery $B_1$.

When the connections are made as indicated in Fig. 20, a current will be established in the circuit in the direction indicated by the arrow. The two cells of the battery $B_1$ will discharge, *i.e.*, the same chemical action will take place in each as takes place in a single cell when its terminals are connected by an external conductor, and electric energy will be transferred from these cells. With the exception of the energy dissipated as heat in the wires $W_1$ and $W_2$, due to their resistance, this energy will be transferred to the cell $B_2$. In the latter cell, a chemical action takes place which is the reverse of that which takes place when this cell is discharging, *i.e.*, the lead sulphate which was formed on its plates is decomposed, and lead peroxide is formed on the positive plate of the cell and spongy lead on the negative plate of the cell, and the specific gravity of the electrolyte increases, due to the formation of sulphuric acid.

This reverse chemical action within the cell $B_2$ represents an increase in the chemical energy of the plates and electrolyte. This increase in chemical energy may be looked upon as due to the transformation within the cell $B_2$ of a part of the electric energy which is transferred to it by the electric current. Due to the resistance of the conductors forming this cell, some of the electric energy transferred to it is also converted into heat. The amount converted into heat, however, is relatively small compared with that which is converted into chemical energy.

As a result of this conversion of electric energy into chemical energy within this cell, it is gradually restored to its original condition, and is therefore said to become "charged." Again, it should be kept in mind that the cell is not charged with electricity, but with energy in a form which may be converted into electric energy by allowing the cell to discharge again. It is common practice, however, to speak of a cell as being given a charge of so many ampere-hours, meaning thereby that the specified number of ampere-hours is caused to flow through the cell from its positive to its negative plate.

A storage cell when discharging is then a source of electric energy, and when charging is a receiver of electric energy. The simple circuit shown in Fig. 20 is typical of the most general electric circuit, namely, a circuit in which there is both a source

of electric energy and a receiver of electric energy, and in which latter there is a transformation of electric energy into both heat energy due to the resistance of the receiver and into some other form of energy. As will be seen later, the electric energy delivered to a receiver may be converted into other forms of energy than chemical energy (*e.g.*, into mechanical energy, when the receiver is a motor) but a part, at least, is always converted into heat energy, due to the resistance of the conductors which form the receiver.

By keeping clearly in mind the energy relations in the concrete case above described, and illustrated by Problems 12 and 13 which follow, the reader should have no difficulty in understanding the definitions given in the next article.

**Problem 12.**—When a storage cell discharges 100 ampere-hours, 550,000 foot-pounds (746,000 joules) of chemical energy are converted into electric energy. (The corresponding chemical change results in the formation within the cell of approximately 1.25 ounces of water and 2.5 pounds of lead sulphate.)

(a) How many watt-hours of electric energy are developed within the cell? (b) If the cell discharges at the rate of 20 amperes, what is the average electric power developed, in watts? (c) If the internal resistance of the cell during discharge averages 0.015 ohm, how many watts are dissipated as heat within the cell? (d) What is the average electric power transferred from the cell? (e) What is the average electric *energy* developed within the cell *per coulomb* of electricity which flows through it? (f) What is the average electric *power* developed within the cell *per ampere* of the current through it? (g) What is the average power per ampere dissipated as heat within the cell? (h) What is the average electric power per ampere transferred from the cell?

*Answer.*—(a) 207 watt-hours. (b) 41.4 watts. (c) 6 watts. (d) 35.4 watts. (e) 2.07 joules per coulomb. (f) 2.07 watts per ampere. (g) 0.3 watts per ampere. (h) 1.77 watts per ampere.

**Problem 13.**—After 100 ampere-hours have been taken from the cell described in Problem 12, the cell is recharged at a constant rate of 20 amperes. If the average internal resistance of the cell when charging remains at the same value (0.015 ohm) as during the discharge and the conversion of electric energy into chemical energy within the cell is at the same average rate, namely, 2.07 watts per ampere, as the reverse transformation during the discharge:[1]

[1]As a matter of fact, both the resistance of the cell and the rate of transformation of energy within it vary with the rate of charge or discharge, and also depend upon whether the cell is being charged or discharged; see Article 72. These variations are due chiefly to variations in the concentration of the electrolyte. As a result of these variations more ampere-hours must be put into a cell to charge it than can be taken from it on discharge.

(a) How many watt-hours of electric energy must be supplied to the terminals of the cell in order to restore the 550,000 foot-pounds of chemical energy lost by the cell during the discharge?  (b) What will be the average electric power input to the terminals of the cell during the charging period?  (c) What will be the average electric power input to the terminals of the cell *per ampere?*

*Answer.*—(a) 237  watt-hours.  (b) 47.4  watts.  (c) 2.37  watts  per ampere.

**38. Electric Pressure and Electromotive Force.**—A review of Article 21 at this point will be found extremely helpful in appreciating the significance of the quantities which will now be defined.

Experiment shows that the electric power developed within a source of electric energy depends in general not only upon the current supplied by this device, but also upon another factor which depends primarily upon the nature of the device, and which in many instances is practically independent of the intensity of the current.  For example, when a storage cell discharges at the rate of 15 amperes, this cell loses, in a given time, approximately 3 times as much energy as is lost by it when it discharges at the rate of 5 amperes for this same length of time.

Exactly the same sort of thing is true of an ordinary centrifugal pump.  The work done by the moving element of the pump depends not only upon the rate of flow of water through it (*i.e.*, upon the "intensity" of the water current through the pump), but also upon the pressure developed by the rotating vanes. When the pump is driven at a constant speed the pressure developed by the vanes is approximately constant, irrespective of the rate at which it delivers water, and therefore, since the pressure developed is proportional to the quotient of the power developed divided by the *rate* of flow of water, this quotient is likewise approximately constant.  Also, since the pressure developed by the rotating vanes depends upon their dimensions and design, the power developed by a given pump per unit rate of flow of water through it may be looked upon as a definite characteristic, or property, of this particular pump.

From such hydraulic analogies, the electric power developed in any source of electric energy *per unit intensity of the electric current* through it, is called the "electric pressure" developed within the device.  That is, calling $p'$ the electric power developed within any source of electric energy, and $i$ the intensity of

the electric current through it, the electric pressure developed within the device is, by definition,

$$e = \frac{p'}{i} \tag{5}$$

The term "electromotive force" (abbreviated e.m.f.), originally introduced to designate the force which causes electricity to move (see Article 2), is also used, in a *quantitative sense*, to designate this quotient. That is, **as the measure of the electromotive force of a source of electric energy is taken the electric power developed within it per unit of current through it.**

The reader should note carefully, however, that electromotive force as thus quantitatively defined is not a force in the ordinary mechanical sense, *i.e.*, is not measurable in dynes or pounds, but is power per unit current, or, what amounts to the same thing, is energy per unit quantity of electricity, or work per unit charge of electricity. (Compare also with the pressure developed by the rotating vanes of a centrifugal pump, which is equal to the work done by the pump per unit quantity of water which flows hrough it; see Article 21.)

Just as a source of electric energy is analogous to a pump, so is a receiver in which electric energy is converted into any form, other than the heat dissipated within it due to its electric resistance, analogous to a hydraulic motor. The flow of water through such a motor transfers energy to it, and, with the exception of the energy which is dissipated as heat due to the resistance of the water passages through the motor, this energy is transferred by the moving element of the motor to whatever is connected to its shaft. The work done by the moving element of the motor represents an equal amount of work done on it by the water, and therefore the moving element must exert on the water a pressure which tends to oppose the flow of the latter. That is, in a water motor there is always a *back* pressure, which is proportional to the power developed by its moving element per unit rate of flow of water through it.

Following this analogy, a receiver in which electric energy is converted into any form, other than the heat which is dissipated within it due to its electric resistance, is said to be the source of an electric pressure, or electromotive force, which *opposes* the flow of electricity through it. Quantitatively, the electromotive force of a receiver may then be defined as the electric power per unit

of current which is converted within the device into any form, other than the heat which is dissipated within it due to its electric resistance. That is, calling $p'$ the electric power which is converted within the device into any form other than the heat which is dissipated within it due to its electric resistance, and $i$ the intensity of the current through it, then the electromotive force of the device is

$$e = \frac{p'}{i} \tag{5a}$$

This equation is of exactly the same form as equation (5), the only difference in the two being that in equation (5) $p'$ represents power of some other form which is converted into electric power, whereas in (5a) $p'$ represents a conversion of electric power into some form of power, other than the heat dissipated due to the internal resistance of the receiver.

The reason for excluding, in the definition of the electromotive force of a receiver, the heat which is dissipated within the receiver due to its electric resistance, is that the power per ampere dissipated as heat varies directly with the current, whereas the power per ampere converted within a receiver into other forms of power is found in many cases to be independent, or approximately independent, of the intensity of the current. Moreover, in the case of a device, such as a storage cell, which may be used either as a source or as a receiver of electric energy, its electromotive force, as above defined, has the same (or approximately the same) value, whether the device acts as a receiver or as a source of electric energy. This relation would not hold were the power dissipated as heat included in $p'$ in equation (5a).

Experiment shows that, in general, when the current through a device is reduced to zero, the quotient of the electric power developed within it, divided by the intensity of the current through it, approaches a definite finite value different from zero. This limiting value of the electromotive force of a device is called its "open-circuit" electromotive force. For example, the open-circuit electromotive force of an ordinary dry cell, when new, is about 1.5 watts per ampere, and the open-circuit electromotive force of a storage cell is about 2.2 watts per ampere. As a rule, the electromotive force of a device falls off somewhat when a current is taken from it.

**39. Units of Electromotive Force.**—By definition, equation (5), an electromotive force is equal to power divided by electric current intensity; or, what amounts to the same thing, to energy divided by quantity of electricity. Hence, in the practical system of units the unit of electromotive force is equal to 1 watt per ampere, or 1 joule per coulomb (since a watt is 1 joule per second and an ampere is 1 coulomb per second). This unit is called the "volt."

Since, from Joule's Law, the power dissipated as heat by 1 ampere in 1 ohm is 1 watt, the electric pressure required to send a current of 1 ampere through a resistance of 1 ohm is equal to 1 watt per ampere, or 1 volt. Hence the following definition of the "international volt" by the London Congress of 1908:

"The international volt[1] is the electrical pressure which, when steadily applied to a conductor the resistance of which is 1 international ohm, will produce a current of 1 international ampere."

As a concrete standard of electromotive force certain special forms of voltaic cells are employed. The standard Clark cell has a constant open-circuit electromotive force of 1.4328 volts at 15°C., and the standard Weston cell has a constant open-circuit electromotive force of 1.01830 volts at 20°C. (see Article 73).

The following figures will give the reader an idea of the magnitude of a volt in terms of more or less familiar apparatus. The open-circuit electromotive force of an ordinary dry cell is about 1.5 volts. A lead storage cell has an open-circuit electromotive force of about 2.2 volts between its terminals. The electric pressure between the two wires of an ordinary house lighting circuit is about 110 volts. The electric pressure between the

[1]The legal definition of the international volt in the United States, as contained in the Act of Congress approved July 12, 1894, is:

"The unit of electromotive force shall be what is known as the internal tional volt, which is the electromotive force that, steadily applied to a conductor whose resistance is 1 international ohm, will produce a current of an international ampere, and is practically equivalent to one thousand fourteen hundred and thirty-fourths of the electromotive force between the poles or electrodes of the voltaic cell known as Clark's cell, at a temperature of fifteen degrees centigrade, and prepared in the manner described in the standard specifications."

Recent experiments give for the exact value of the electromotive force of the Clark cell, under the conditions specified, the figure of 1.4328 international volts, based upon the definitions of the London Congress.

trolley wire and the track rails of most direct-current railways is from 500 to 600 volts. For the transmission of large amounts of energy over long distances electric pressures of from 2000 to 150,000 volts are used.

In the practical system of units, very small electric pressures are usually expressed in millivolts, a millivolt being 0.001 volt, and very high electric pressures are expressed in kilovolts, a kilovolt being 1000 volts.

In the c.g.s. electromagnetic system of units the unit of electromotive force, called the abvolt, is equal to 1 erg per second per abampere, or to 1 erg per abcoulomb. In the c.g.s. electrostatic system the unit of electromotive force, called the statvolt, is 1 erg per statcoulomb. These units are related as follows:

$$1 \text{ volt} \quad = 10^8 \text{ abvolts.}$$
$$1 \text{ statvolt} = 300 \text{ volts.}$$
$$1 \text{ statvolt} = 3 \times 10^{10} \text{ abvolts.}$$

In terms of the international volt and the international ampere, the international watt is defined by the London Congress as "the energy expended per second by an unvarying electric current of 1 international ampere under an electric pressure of 1 international volt." The international watt as thus defined is equal to 1.00044 "absolute" watts, an absolute watt being by definition $10^7$ ergs per second (compare Article 36).

**Problem 14.**—(a) What is the value of the electromotive force of the storage cell described in Problem 12 when the cell is discharging at the rate of 20 amperes? (b) If all the electric power developed in the cell is converted into heat energy in the conductors which form the path of this current, what is the total resistance of the circuit of which the cell forms a part? (c) What is the resistance of the portion of this circuit external to the cell? (d) What is the ratio of the electric power transferred from the cell to the current in the external circuit? (e) What is the difference between this ratio and the electric power per unit current developed in the cell? (f) Compare this difference with the product of the internal resistance of the cell by the current through it. (g) Compare the electric power transferred from the cell per ampere with the product of the resistance of the external portion of the circuit by the current through it.

*Answer.*—(a) 2.07 volts. (b) 0.1035 ohms. (c) 0.0885 ohms. (d) 1.77 volts. (e) The electric power transferred from the cell is 0.3 volts less than the electromotive force of the cell. (f) This difference is equal to the product of the current through the cell by its internal resistance. (g) The electric power transferred from the cell per ampere is equal to the product of the resistance of the external portion of the circuit by the current through it.

**Problem 15.**—(a) What is the value of the electromotive force of the storage cell described in Problem 13 when charging, under the same assumptions as in Problem 13? (b) What is the ratio of the total electric power input to this cell to the current through it? (c) What is the difference between this ratio and the electromotive force of the cell? (d) Compare this difference with the product of the internal resistance of the cell by the current through it.

*Answer.*—(a) 2.07 volts. (b) 2.37 volts. (c) The electric power input per ampere is 0.3 volts greater than the electromotive force of the cell. (d) This difference is equal to the product of the current through the cell by its internal resistance.

**40. Difference of Electric Potential and Resistance Drop.**— Every source or receiver of electric energy consists of one or more conductors, and therefore the flow of electricity through either kind of device is in general accompanied by a dissipation of electric energy as heat energy within the device, due to the electric resistance of the conductors of which it consists. Hence, in the case of a source of electric energy the electric power transferred from the device, *i.e.*, the *net* electric power output, is always less than the electric power developed within it. Similarly, in the case of a receiver of electric energy, the electric power input to the device is always greater than the electric power which is converted within it into some other form than the heat which is dissipated within it due to its resistance.

Hence, for a source of electric energy the net electric power output per ampere is always less than its electromotive force, and for a receiver of electric energy the electric power input per ampere is always greater than its electromotive force. Compare with a pump and hydraulic motor respectively (Article 21).

The net electric power *output* from a source of electromotive force, per unit intensity of the current which flows through the device from one terminal to the other, is called the "terminal" electric pressure of the device, or the "difference of electric potential" between these terminals. These same terms are also used to designate the electric power *input* to a receiver of electric energy, per unit intensity of the current through it from one terminal to the other. In the case of a receiver the electric power input per unit of current is also called the electric pressure "impressed" across its terminals. Electric potential difference is frequently abbreviated "p.d."

Terminal electric pressure, or electric potential difference, has the same "dimensions" as electromotive force, namely, power

divided by current intensity, or energy divided by quantity of electricity, and is therefore expressed in the same units, viz., volts, abvolts or statvolts.

Electric potential difference when expressed in volts is commonly referred to as the "voltage." For example, the potential difference between the terminals of a device, when expressed in volts, is called the "terminal voltage," or in the case of a receiver, the "impressed voltage." Similarly, the electromotive force of a device when expressed in volts is frequently referred to as the "internal voltage," or, in the case of a source of electric energy, as the "generated voltage."

The terms "impressed electromotive force" and "terminal electromotive force" are also used to designate the difference of potential between the terminals of a device. The terminal electromotive force of a device, however, must not be confused with its true, or "internal," electromotive force. For convenience in distinguishing between terminal electric pressure (or electric potential difference) and the internal electric pressure (or electromotive force), the former will be designated throughout this book by the symbol $v$ or $V$ and the latter by the symbol $e$ or $E$.

When the current $i$ through a *source* of electric energy has but a single path of resistance $r$, a very simple relation exists between the electromotive force $e$ of the device and the difference of electric potential between its terminals. From the definition of electromotive force, the total electric power developed within the device when the current $i$ flows through it, is $ei$; compare equation (5). Of this total power developed the amount $ri^2$ is dissipated as heat; compare equation (4). The net electric power output is (by the definition of the potential difference $v$) equal to $vi$, and this must be equal to the total electric power developed less that dissipated as heat, viz., $vi = ei - ri^2$. Dividing this equality by $i$ gives

$$v = e - ri \qquad (6)$$

The same considerations applied to a *receiver* of electric energy in which the current $i$ has a single path of resistance $r$, shows that the difference of potential $v$ between the terminals of a receiver is equal to its electromotive force $e$ *plus* the product of the current by the internal resistance of the receiver, viz.,

$$v = e + ri \qquad (6a)$$

In the case of a receiver of electric energy *in which there is no electromotive force, i.e.*, in which all the electric power supplied to it is dissipated as heat due to its resistance $r$, equation (6a) reduces to

$$v = ri \tag{6b}$$

which may also be written

$$i = \frac{v}{r} \tag{6c}$$

That is, for any receiver in which electric energy is converted into *heat energy only*, due *solely* to the electric resistance of the path of the current through it, the difference of potential between its terminals is equal to the product of its resistance by the intensity of the current through it.

**Hence, for a receiver in which there is no electromotive force and which has a constant resistance, the difference of potential between its terminals is directly proportional to the intensity of the current through it.** Conversely, the intensity of the current through such a receiver is directly proportional to the value of the potential difference impressed across its terminals. This relation, of which equation (6b) or (6c) is the mathematical expression, is known as "Ohm's Law," from the name of the scientist who first discovered it.

The relations expressed by equations (6) and (6a) are of exactly the same form as the corresponding relations (equations (25) and (24) of Article 21) for a pump and hydraulic motor respectively, the product $ri$ in equations (6) and (6a) corresponding to the drop of pressure $V_r$ due to the frictional resistance of the water passages of the .pump or motor.

The product of the resistance of any portion of the path of an electric current by the intensity of this current always represents a potential difference, whether this portion of the path contains an electromotive force or not. This product, viz., $ri$, is commonly referred to as the drop of potential due to the electric resistance of this portion of the path of the current, or briefly, as the "resistance drop" in this portion of the path.

A resistance drop is also fréquently referred to as the $IR$ drop (pronounced $I - R$ drop), and the power dissipated as heat in a device due to its electric resistance as the $I^2R$ loss (pronounced $I$ squared $R$ loss).

A brief statement of the facts represented by equations (6) and (6a), viz., by the equations,

(6) $\qquad v = e - ri \qquad$ for a source of electric energy

(6a) $\qquad v = e + ri \qquad$ for a receiver of electric energy

is then that **the potential difference between the terminals of a source of electric energy is less than its internal electromotive force by an amount equal to the resistance drop through it, and the potential difference between the terminals of a receiver is greater than its internal electromotive force by an amount equal to the resistance drop through it.** This relation, of which equations (6) and (6a) are the mathematical expression, is sometimes referred to as the "Generalized Ohm's Law."

From equation (6) it follows that when there is no current through a source of electromotive force, the potential difference
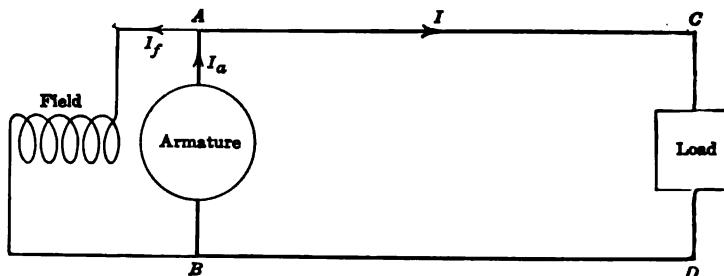


FIG. 21.—Shunt generator and load.

between its terminals is equal to its electromotive force. Similarly, from (6a), in the case of a receiver through which there is no flow of electricity, the potential difference between its terminals is equal to its electromotive force. Hence, the "open-circuit" electromotive force of any device is the same as the potential difference between its terminals when there is no flow of electricity through it.

**Problem 16.**—Referring to Problems 12 and 13, what is the terminal voltage of the storage cell there described:

(a) When discharging at the rate of 20 amperes? (b) When charging at the rate of 20 amperes? (c) What is the resistance drop through the cell in each case?

*Answer.*—(a) 1.77 volts. (b) 2.37 volts. (c) 0.3 volt. (NOTE.—As pointed out in the footnote to Problem 13, the internal resistance, and therefore the internal resistance drop, will actually vary during charge and discharge due to changes in the concentration of the electrolyte.)

**Problem 17.**—A direct-current "shunt-wound" dynamo consists essentially of two sets of coils of insulated wire wound on iron cores. One set of coils and cores, called the "field," is stationary, and the other set of coils and core, called the "armature," is caused to rotate, being driven by an engine or motor of some kind. In virtue of the electromagnetic reactions between the armature and field windings, the mechanical power supplied by the driving engine is converted within the armature winding into electric energy. A diagrammatic sketch of the two windings is shown in Fig. 21, in which are also shown a receiver, or external "load," and a transmission line (two parallel wires) connecting this load to the generator. When the total current ($I_a$) supplied by the armature is 325 amperes, 100 horsepower of mechanical power is converted within the armature into electric power. The internal resistance of the armature between the two terminals $A$ and $B$ is 0.02 ohm, and the resistance of the field circuit between these same two terminals is 20 ohms. All the electric power supplied to the field circuit is converted into heat.

(a) What is the electromotive force generated in the armature winding? (b) What is the resistance drop through the armature between the terminals $A$ and $B$? (c) What is the difference of potential between the armature terminals? (d) What is the value of the current $I_f$ in the field circuit? (e) How much electric power is supplied to the field circuit? (f) What is the value of the current ($I$) in each wire of the transmission line? (g) What is the value of the current through the load? (h) What is the electric power output of the generator to the transmission line? (i) If the resistance of each of the two line wires is 0.03 ohms, how much electric power is dissipated as heat in these two wires? (j) How much electric power is supplied to the load? (k) What is the potential difference between the terminals of the load?

*Answer.*—(a) 229.5 volts. (b) 6.5 volts. (c) 223.0 volts. (d) 11.15 amperes. (e) 2.486 kilowatts. (f) 313.9 amperes. (g) 313.9 amperes. (h) 70.00 kilowatts. (i) 5.912 kilowatts. (j) 64.09 kilowatts. (k) 204.1 volts.

**Problem 18.**—If the "load" in Problem 17 is another shunt-wound dynamo, identical in design with that used as the generator, but operating as a motor (*i.e.*, converting electric energy into mechanical energy):

(a) What will be the current in the field circuit of this motor? (b) What will be the current through its armature? (c) What will be the resistance drop in its armature? (d) What will be the electromotive force in this armature? (e) How much mechanical power will be developed by this armature? (f) What percentage of the total power which is converted from mechanical to electric power in the generator is converted into mechanical power in the motor?

*Answer.*—(a) 10.20 amperes. (b) 303.7 amperes. (c) 6.074 volts. (d) 198.0 volts. (e) 60.13 kilowatts or 80.60 horsepower. (f) 80.60 per cent. (NOTE.—In both a generator and in a motor there are in addition to the $RI^2$ losses other losses, due to mechanical friction and to energy dissipated in the iron. Hence, more power must be developed by the engine than is converted into electric power in the generator, and the mechanical power avail-

able at the motor pulley is less than that converted into mechanical power in the motor. The overall efficiency from the engine shaft to the motor pulley under the conditions specified in this and the preceding problem is therefore actually less than 80.60 per cent.)

**41. Direction of a Potential Difference, of an Electromotive Force, and of a Resistance Drop.**—The absolute pressure at the outlet of a pump is always greater than the absolute pressure at its intake, and the absolute pressure at the intake of a hydraulic motor is always greater than the absolute pressure at its outlet. From the similarity between hydraulic pressure and electric pressure, or electric potential difference, the electric potential at that terminal of a *source* of electric energy at which the current *leaves* may be said to be higher than the electric potential at the terminal at which the current enters. On the basis of this convention, the electric potential at that terminal of a *receiver* at which the current *enters* must then be higher than the electric potential at that terminal from which the current leaves.

In accord with this definition, then, through a *source* of electric energy there is always a *rise* of electric potential *in the direction of the current*, and through a *receiver* there is always a *drop* of electric potential *in the direction of the current* (compare with the pump and hydraulic motor in Article 21). That terminal of a device which is at the higher potential is called the positive terminal, and that terminal which is at lower potential is called the negative terminal. The positive terminal of a *source* is then always the terminal from which the current *leaves*, and the positive terminal of a *receiver* is always that terminal at which the current *enters*.

The direction of the electromotive force in any device is taken as the direction through it from its negative to its positive terminal. Hence, in a source of electric energy the electromotive force and the current are always in the same direction, whereas in a receiver the electromotive force and current are always in opposite directions. This is in accord with the qualitative definition of an electromotive force of a source of electric energy as that which produces, or tends to produce, a flow of electricity in the circuit of which it is a part, and the qualitative definition of the electromotive force of a receiver as that which, other than its resistance, opposes the flow of electricity through it.

Since the electromotive force of a receiver is in the opposite direction to that of the current through it, its electromotive force

is frequently referred to as a "back," or "counter," electromotive force.

The conductors forming the path of a current through any device are always receivers of electric energy to the extent to which heat energy is dissipated within them due to their electric resistance. Hence the product of the resistance of a device by the current through it, *i.e.*, the product $ri$ always represents a drop of electric potential through the device in the direction of the current through it, or a rise of potential through the device in the direction opposite to that of the current through it.

The fundamental equation (6) for a source of electric energy, viz.,

$$v = e - ri$$

may then be read: The net drop of potential from the positive to the negative terminal of a source of electric energy through which a current is flowing is equal to the rise of potential through it due to its electromotive force, less the drop of potential through it due to its resistance.

Similarly, the fundamental equation (6a) for a receiver of electric energy, viz.,

$$v = e + ri$$

may be read: The net drop of potential from the positive to the negative terminal of a receiver of electric energy through which a current is flowing is equal to the rise of potential through it due to its electromotive force, plus the drop of potential through it due to its resistance.

**Problem 19.**—Referring to Problem 17:

(a) Which is the positive and which the negative terminal of the generator? (b) Which is the positive and which the negative terminal of the load? (c) In which direction is the drop of potential in the wire $AC$? (d) In the wire $BD$? (e) If the load is a motor, what is the direction of the electromotive force of the motor? (f) Draw a diagram showing the two windings of the motor, the transmission line and the two windings of the generator, and indicate by means of an arrow alongside of each winding the direction of the electromotive force in this winding; mark these arrows $E$, with a suitable subscript in each case. (g) Show also by an arrow alongside of each winding and line wire the direction of the current in each; mark these arrows $I$, with a suitable subscript in each case.

*Answer.*—(a) $A$ is positive and $B$ negative. (b) $C$ is positive and $D$ negative. (c) From $A$ to $C$. (d) From $D$ to $B$.

6

**42. Kirchhoff's Second Law for Networks.**—Any network of conductors, such as shown in Fig. 22, ho matter how complex, may always be considered as made up of individual sources and receivers of electric energy through each of which the current has the same value from one terminal to the other. The definition of electric potential difference given in Article 40 may be extended to apply to any two points in such a network, whether or not they be the terminals of the same source or receiver of electric energy, by defining the difference of electric potential between any two points of such a network as the *algebraic* sum of the potential drops in the several branches through which a
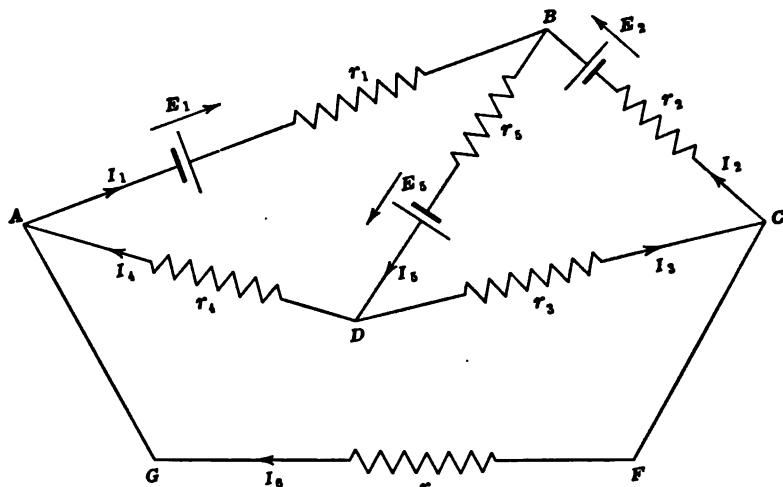


Fig. 22.

point must move in order to pass continuously from one of the given points to the other. In this definition an actual rise of potential in any branch in any given direction is considered as equivalent to a *negative drop* of potential in this same direction.

Experiment shows that the difference of electric potential between two points, as thus defined, is independent of the path over which the moving point is assumed to pass from one fixed point to the other. This fact, namely, that the difference of electric potential between any two given points has at any instant one and only one value, and is equal to the algebraic sum of the potential drops in any path from one point to the other, is one of the fundamental "laws" of electricity. This law is known as

Kirchhoff's Second Law for Conducting Networks (see Article 29 for Kirchhoff's First Law).

In order to illustrate the significance of this law, consider the network shown in Fig. 22. If the electromotive forces and currents in this network are actually in the directions indicated by the arrows, and a point is imagined to move from $A$ to $C$ through the branches $AB$ and $BC$, the potential of this moving point with respect to the potential of $A$ will increase by an amount $E_1$ as the point moves through the electromotive force $E_1$, will decrease by an amount $r_1I_1$ as the point moves through the resistance $r_1$, will decrease by the amount $E_2$ as the point moves through the electromotive force $E_2$, and will increase by the amount $r_2I_2$ as the point moves through the resistance $r_2$. The resultant decrease, or drop, in electric potential from $A$ to $C$ is then

$$V_{AC} = -E_1 + r_1I_1 + E_2 - r_2I_2$$

Again a point may move from $A$ to $C$ by passing through the branches $AD$ and $DC$. The drop of potential from $A$ to $C$ is therefore also

$$V_{AC} = -r_4I_4 + r_3I_3$$

and similarly for any other path from $A$ to $C$.

A general mathematical statement of Kirchhoff's Second Law is that the drop of electric potential from any point $A$ to any other point $B$ is

$$v_{AC} = \Sigma ri - \Sigma e \qquad (7)$$

where $\Sigma ri$ represents the algebraic sum of the resistance drops in any path from $A$ to $B$, and $\Sigma e$ is the algebraic sum of the electromotive forces in this same path in the direction from $A$ to $B$.

This law may be stated in other ways (see Chapter III), but for the present the fundamental fact to keep in mind is that between two given points the electric potential difference has at any instant one and only one value, irrespective of the number of branches connecting these points, and irrespective of the resistances and of the electromotive forces of the several branches.

The fact that the difference of electric potential between two points has one and only one value is another instance of the analogy between electric potential and hydraulic pressure. Just as at each point in a reservoir or system of pipes filled with water, the absolute pressure has at any instant one and only

one value, so also is each point in any system of conductors characterized at each instant by a single-valued property, called its electric potential. For the present, it is not necessary to define quantitatively the "absolute" value of the electric potential at a point, for in any problem involving the generation, transmission or utilization of electric energy, it is always the *difference* of electric potential between two points which has to be considered, and not the absolute values of the potentials at the two points.

Problem 20.—In the network represented by Fig. 22 the several resistances, electromotive forces and currents have the values:

| | | |
|---|---|---|
| $E_1 = 10$ volts, | $r_1 = 1$ ohm, | $I_1 = 2$ amperes, |
| $E_2 = 14$ volts, | $r_2 = 4$ ohms, | $I_2 = 4$ amperes, |
| | $r_3 = 3$ ohms, | $I_3 = 5$ amperes, |
| | $r_4 = 0.5$ ohm, | |

and the electromotive forces $E_1$ and $E_2$ and the currents $I_1$, $I_2$ and $I_3$ have the directions indicated by the arrows. Determine:

(a) The value and direction of the current in branch No. 5. (b) The value and direction of the current in branch No. 4. (c) The value and direction of the current in branch No. 6. (d) The value and direction of the potential difference between $A$ and $D$. (e) The value and direction of the potential difference between $A$ and $C$. (f) The value and direction of the potential difference between $D$ and $C$. (g) The value and direction of the electromotive force in branch No. 5. (h) The resistance of branch No. 4. (i) The resistance of branch No. 6.

*Answer.*—(a) 6 amperes from $B$ to $D$. (b) 1 ampere from $D$ to $A$. (c) 1 ampere from $C$ to $A$. (d) 25 volts drop from $D$ to $A$. (e) 10 volts drop from $C$ to $A$. (f) 15 volts drop from $D$ to $C$. (g) 20 volts from $B$ to $D$. (h) 25 ohms. (i) 10 ohms.

**43. The Voltmeter.**—As a consequence of the fundamental principle that the difference of electric potential between two points can have but a single value, it follows that when a conductor is connected between any two points $A$ and $B$ of any circuit whatever, the drop of potential from a point $A$ to the point $B$ through this conductor will be exactly equal to the drop of potential from the point $A$ to the point $B$ through the conductors of the given circuit.

From Ohm's Law, equation (6b), the potential drop through a conductor in which there is no electromotive force is equal to the product of the resistance of this conductor by the intensity of the current through it. Hence, by connecting the two points in question with a conductor of known resistance $r$, in which there is

no electromotive force, and measuring the current $I$ through this conductor, the potential difference $rI$ between the two points may be determined.   This is the principle of the ordinary form of voltage measuring instrument, or electromagnetic voltmeter.

In construction, the electromagnetic type of voltmeter is practically identical with the corresponding type of ammeter, except that a high resistance is connected in series with the winding on the moving element, and the scale of the instrument is calibrated to read directly in volts instead of amperes.   (This high resistance is made of an alloy having a high resistivity and practically negligible resistance-temperature coefficient, with the result that the indication of the instrument is practically independent of temperature; see Article 62.)

It should be particularly noted that unless the resistance of a voltmeter is relatively very large, the current taken by it may by sufficient to produce an appreciable change in the current in the main circuit, and therefore also in the original difference of potential between the two points to which it is connected.   Also, it should be noted that the contact electromotive forces (see Article 70) at the junctions between the voltmeter leads and the circuit to which it is connected may, when the leads and the circuit are of different materials, cause an error in the reading of the instrument. In all ordinary measurements, however, this latter effect is negligible.

**44. Millivoltmeter and Shunt for Measuring Large Currents.—** The current which can be sent through the movable element of the ordinary moving-coil ammeter is limited to about $\frac{1}{20}$ ampere, due to the fact that the springs necessary to carry a large current without overheating would be so large that their torsion would seriously affect the deflection of this element.   In order to employ this type of instrument for measuring larger currents, it is necessary to "shunt" the major part of the current through a resistance which is connected in parallel with the movable element, as shown in Fig. 23.   This is accomplished by designing the shunt so that it has a relatively low resistance (*i.e.*, by making it of a relatively large cross-section), and by designing the winding of the movable coil so that it has relatively a very high resistance (*i.e.*, by making it a coil of a number of turns of fine wire).   In order to decrease still further the current which passes through the movable coil, a fixed coil of fine wire is also usually·inserted in

series with it.  The deflection of the movable coil then actually measures the *potential drop* through the shunt.

The shunt may be contained within the case of the instrument or may be external to it as shown in the figure.  The resistance of the shunt is usually such that the potential drop through it for the maximum current with which it is to be used is from 50 to 100 millivolts.  The movable coil and the fixed resistance then constitute a voltmeter having this same range.  When the shunt is separate from the instrument, the instrument itself is called a millivoltmeter.  Its scale in this case is usually marked to read directly in millivolts, and a proper multiplier is marked on the shunt whereby the indications of the instrument may be converted into amperes.
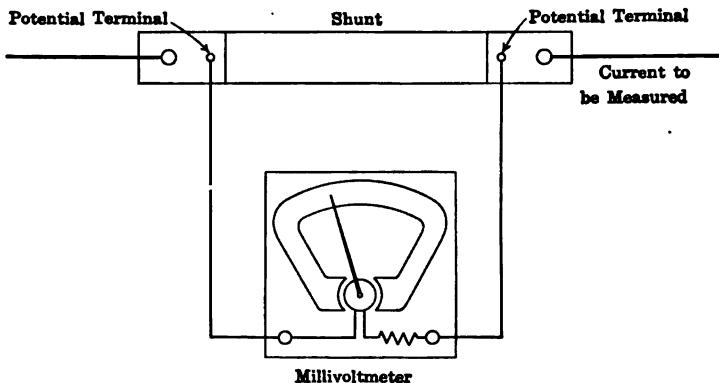


FIG. 23.—Millivoltmeter and shunt.

The two ends of the shunt are usually made of heavy pieces of copper, between which are connected one or more strips of a high-resistance alloy.  The major portion of the potential drop through the shunt is in these latter strips.  Separate "potential" terminals, at a distance from the "current" terminals, are used for making the connection to the millivoltmeter, in order to avoid the possibility of error due to the contact resistance at the current terminals.

The exact relation between the reading of a millivoltmeter and the total current in the line in which the shunt is connected, may be readily deduced from the fundamental principle that between two points the difference of potential at any instant has but one value.  Let $V$ be this reading in millivolts, $r$, the resistance, in

ohms, between the two potential terminals of the shunt, and $r_v$ the resistance, in ohms, of the millivoltmeter and the leads between it and the shunt. Then the current through the shunt is $\dfrac{V}{1000\,r_s}$, and the current through the instrument is $\dfrac{V}{1000\,r_v}$. The total current in the line is then

$$I = \frac{V}{1000}\left(\frac{1}{r_s} + \frac{1}{r_v}\right)$$

$$= \frac{V}{1000\,r_s}\left(1 + \frac{r_s}{r_v}\right) \tag{8}$$

When the resistance of the shunt is negligible in comparison with the resistance of the millivoltmeter, the multiplier is then $\dfrac{1}{1000\,r_s}$, but when the resistance of the shunt is appreciable in comparison with the resistance of the millivoltmeter, say 0.1 per cent. of the latter or greater, the multiplier must be increased by this same percentage.

In particular it should be noted that when a low-resistance millivoltmeter is used with a shunt, the resistance of the leads which connect the instrument to the shunt may produce an appreciable effect. For precise work the multiplier should be determined for a given pair of leads, and these leads and no others should be used.

**45. The Potentiometer.**—A voltmeter may be readily calibrated in terms of the electromotive force of a standard cell by employing an arrangement of circuits known as a "potentiometer." This apparatus may also be used to measure directly the difference of potential between two points, with a degree of precision of one part in 10,000, or better. The potentiometer method, like all other methods of comparing potential differences, is based upon the fundamental principle that between two points the potential difference at any instant can have but a single value.

Referring to Fig. 24, $B$ is a battery or other source of electromotive force which is connected to the two ends of a wire of uniform cross-section. To the end $A$ of this wire is also connected a standard cell $C$, say a Weston cell, in series with a galvanometer $G$, a high resistance and a key $K$. The high resistance in series with the galvanometer is to prevent a large current from flowing through the cell while the adjustments are being made. The other end of this circuit is an adjustable contact $S$ which can be

moved along the wire. The *like* terminals of the two batteries must be connected to the *same* end of the wire, and the electromotive force of the battery $B$ must be greater than the electromotive force of the standard cell.

The contact $S$ is moved along the wire until the galvanometer shows no deflection when the key $K$ is closed. (In the final adjustment the high resistance is to be short-circuited.) Let $r_s$ be the resistance of the wire between $A$ and $S$, $I$ the current in this wire, and $E_s$ the electromotive force of the standard cell. When there is no current through the galvanometer $G$ there will be no current in the circuit from $A$ to $S$ through the galvanometer, and therefore the electromotive force $E_s$ in this branch must be
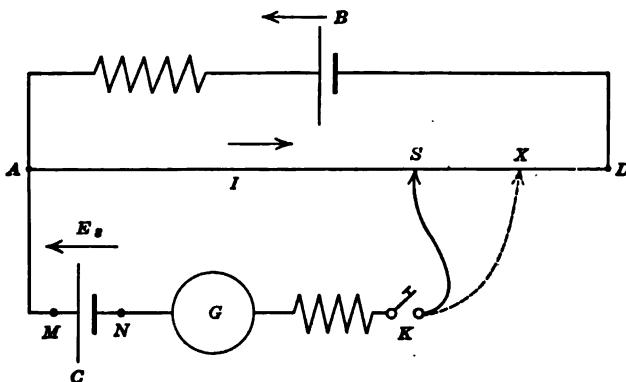


FIG. 24.—Potentiometer.

equal to the resistance drop $r_s I$ in the wire from $A$ to $S$, viz., $E_s = r_s I$.

The standard cell is then removed and the points $M$ and $N$ of the galvanometer circuit are connected to the two points between which the unknown potential difference $V$ exists, and the sliding contact is moved until the galvanometer again shows no deflection when the key $K$ is closed, the current $I$ being kept constant. Let $X$ designate this new position of the sliding contact. The resistance drop $r_x I$ in the length of wire between $A$ and $X$ will then be equal to $V$, viz., $V = r_x I$. But from the first relation $I = \dfrac{E_s}{r_s}$. Whence, for a constant current through the wire,

$$V = \frac{r_x}{r_s} E_s. \tag{9}$$

The resistance per unit length of a given wire of uniform cross-section is a constant for any given temperature (see Article 55). Hence, if the wire $AD$ is of uniform cross-section and the temperature remains the same during both measurements, the unknown potential difference is also

$$V = \frac{l_x}{l_s} E_s \qquad (9a)$$

where $l_x$ and $l_s$ represent respectively the lengths $AX$ and $AS$.

The potentiometer may also be used for comparing two resistances, say an unknown and a known resistance. It is only necessary to connect the two resistances in series and send a constant current through them, and to measure, as just described, the potential drop through each. Let $r_1$ and $r_2$ be the two resistances, $I$ the current through each, and $V_1$ and $V_2$ the potential drops through $r_1$ and $r_2$ respectively. Then $V_1 = r_1 I$ and $V_2 = r_2 I$; whence

$$r_2 = r_1 \frac{V_2}{V_1} \qquad (10)$$

Or, letting $l_1$ and $l_2$ be the lengths of the potentiometer wire corresponding to the two settings of the sliding contact,

$$r_2 = r_1 \frac{l_2}{l_1} \qquad (10a)$$

For further details in regard to the use of the potentiometer and its various applications, the reader is referred to any standard text-book on electrical measurements.

**46. The Wheatstone Bridge.**—A more convenient arrangement of circuits for comparing two resistances, and one which is capable of as high a degree of precision as the potentiometer, provided neither of the resistances is relatively small compared with the "contact" resistance at the junction points, is the so-called "Wheatstone bridge." The principle here involved is likewise the fundamental one that between two points there can exist at any instant but one value of the potential difference.

The arrangement of circuits is shown in Fig. 25. $B$ is a battery of any kind, $G$ a galvanometer, and $r_1$, $r_2$, $r_3$ and $r_4$ are the resistances of the branches between the points $A$ and $C$, $C$ and $D$, $A$ and $F$, and $F$ and $D$ respectively. By varying any one or more of these four resistances the current through the galvanometer may be made zero, which will be indicated by the galvanom-

eter showing no deflection when the keys $K_1$ and $K$ are closed. When this condition is established there will be no resistance drop from $F$ to $C$, and therefore $C$ and $F$ will be at the same potential. Hence, the resistance drop from $A$ to $C$ will be equal to that from $A$ to $F$, and the resistance drop from $C$ to $D$ will be equal to that from $F$ to $D$, namely,

$$r_1 I_1 = r_2 I_2$$
$$r_2 I_2 = r_4 I_4$$

But since there is no current through the galvanometer, $I_1 = I_2$ and $I_3 = I_4$. Whence,

$$\frac{r_1}{r_2} = \frac{r_3}{r_4}$$
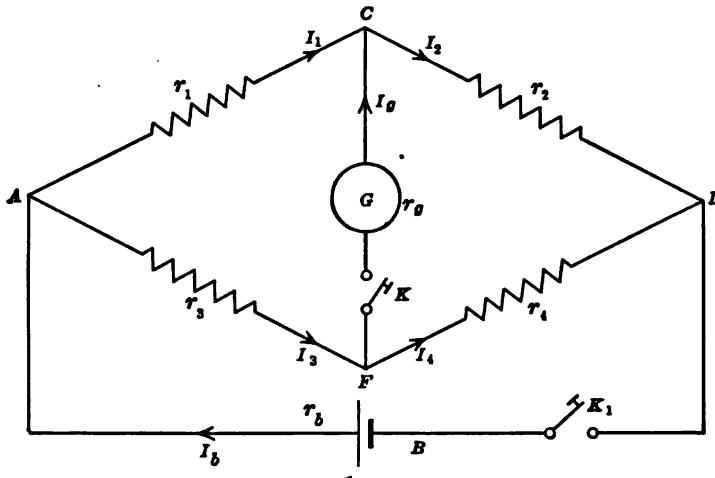


FIG. 25.—Wheatstone bridge.

Hence, when the ratio of the two resistances $r_1$ and $r_2$ is known, and the resistance $r_3$ is changed until there is no current in the galvanometer, which will be indicated by the galvanometer showing no deflection when the keys are closed, the resistance $r_4$ can be calculated from the relation

$$r_4 = \frac{r_2}{r_1} r_3 \tag{11}$$

In the simplest form of Wheatstone bridge, the resistances $r_1$ and $r_2$ are formed by a continuous wire of uniform cross-section, and the resistance $r_3$ is a single standard resistance coil,

or several such coils in series. Instead of altering the resistance $r_3$, the galvanometer terminal $C$ is moved along the wire until the galvanometer deflection becomes zero. The ratio of the two resistances $r_1$ and $r_2$ is then equal to the ratio of the lengths of this wire between the points $A$ and $C$ and between $C$ and $D$ respectively. Call these lengths $l_1$ and $l_2$ respectively; then equation (11) may be written

$$r_4 = \frac{l_2}{l_1} r_3 \qquad (11a)$$

By employing this so-called "slide-wire" type of Wheatstone bridge, it is therefore possible to determine by a very simple experiment the resistance of any conductor in terms of a single standard resistance.

**47. The Wattmeter.**—In the case of direct currents, the electric power input to, or output from, any portion of a circuit may be readily determined by measuring by means of an ammeter, the current $I$ in this circuit,[1] and by measuring by means of a voltmeter, the difference of potential $V$ between the terminals of this circuit. The power is then the product $VI$. When the current is measured in amperes and the potential difference in volts, this power is in watts.

Instead of measuring the current and the potential difference separately by two different instruments, it is possible to measure the value of the product $VI$ directly by means of a single instrument. Such an instrument is called a "wattmeter." The usual form of wattmeter is identical in principle with that of an electrodynamometer (see Article 33), and may be used to measure the average power, irrespective of whether the current and potential difference are constant or vary with time.

When an electrodynamometer is designed for use as a wattmeter, the stationary coil $C_i$ in Fig. 26, is made of a small number of turns of relatively large wire, and the movable coil, $C_v$, is made of a large number of turns of relatively fine wire, and a high resistance $R$ is connected in series with this latter coil. The instrument is connected to the circuit in which the power is to be measured as shown in Fig. 26, with the coil $C_i$ in series with the load, and with the circuit formed by the coil $C_v$ and the resistance $R$ "shunted" across the load (*i.e.*, across the portion

[1] The word "circuit," although meaning strictly a closed loop, is commonly used to designate any specified portion of a circuit.

of the circuit to which, or from which, power is supplied). The coil $C_i$ is therefore called the "current coil," and the coil $C_v$ and the resistance $R$ together is called the "potential circuit" of the instrument.

An inspection of Fig. 26 will show that the current through the current coil $C_i$ is equal to the current taken by the load, and the current through the potential coil $C_v$ is proportional to the difference of potential between the points $A'$ and $B$ to which the potential circuit is connected. As shown in Chapter XI, the torque (*i.e.*, mechanical moment) between two coils carrying currents of different intensities is proportional to the products of these two intensities (provided the reluctance of the magnetic circuit of the instrument is constant, see Article 88). Hence,
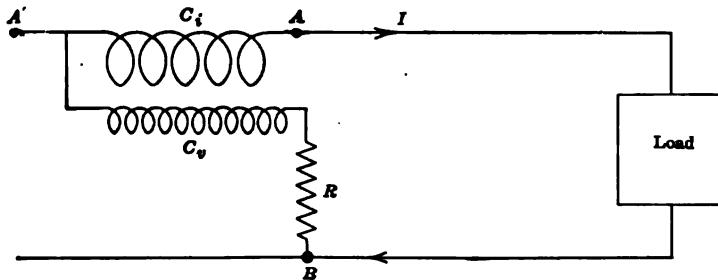


FIG. 26.—Wattmeter connected to load.

the torque tending to deflect the movable element of the instrument is proportional to the product of the voltage across its potential circuit, and the current through the current coil. Neglecting the voltage drop in the current coil (from $A$ to $A'$ in the figure), which is usually permissible, the torque acting on the movable element is then proportional to the power supplied to the terminals $A$ and $B$ of the load.

If the opposing torque produced by the suspension of the movable element, or by a suitable spring attached to it, is proportional to the angular twist given the coil, its deflection will then be proportional to the power transferred to the circuit between these two points. A suitable pointer attached to the movable element and arranged to move over a suitably graduated scale will then indicate directly the value of the power.

## DIRECT-CURRENT CIRCUIT CALCULATIONS

**48. Fundamental Principles Involved in Circuit Calculations.** —An electric device usually contains two or more conductors in series or in parallel, or in series-parallel, and the device as a whole may form part of a more or less complex network of conductors and other devices.

For example, a voltaic cell consists of two dissimilar metals and an electrolyte in series. Again, in a dynamo (see Problem 17, Chapter II) the armature and field windings may be connected in parallel or in series, or a part only of one winding may be in parallel with the other.

Again, in a power station it is common practice to connect two or more generators in parallel. This is usually done by connecting the positive terminals of the several generators, through suitable switches, to a common conducting bar, called the positive "busbar," and the negative terminals in a similar manner to the negative busbar. A number of transmission lines, or feeders, may be run out from these busbars, and each transmission line may supply several loads connected to it at different points. The various feeders from one or more power stations, or substations, may also be interconnected at various points, forming networks of a greater or less degree of complexity.

The calculation of the currents in the several circuits of a machine, or in the several branches of a network, when the electromotive force in, and the resistance of, each branch is known, is therefore one of the commonest problems in electrical engineering. But two fundamental principles are involved in such calculations, namely, (1) the total current coming up to any junction point $A$ is equal at each instant to the current leaving that point, provided there is no change in the electric charge at that point (see Article 29), and (2) the difference of electrical potential between two given points $A$ and $B$ at any instant is equal to the *algebraic* sum of the resistance drops in any path from $A$ to $B$, *less* the *algebraic* sum of the electromotive forces

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in this same path in the direction from $A$ to $B$. As already noted (Articles 29 and 42), these two principles are known as Kirchhoff's First and Second Laws for Conducting Networks.

Throughout this chapter it will be assumed that there is no change in the charge on any part of the network, which assumption is strictly true in the case of direct currents, *i.e.*, currents which do not vary with time (see Article 29), and is substantially true for variable currents, except for junctions which are extended surfaces with a conductor on one side and a dielectric on the other. The two fundamental principles just stated may then be expressed mathematically,

$$\Sigma i_A = 0 \tag{1}$$

$$v_{AB} = \Sigma ri - \Sigma e \tag{2}$$

In the first equation $\Sigma i_A$ represents the *algebraic* sum of the currents *entering* any junction point $A$, a current being considered as negative if its actual direction is away from $A$. In the second equation $\Sigma ri$ represents the algebraic sum of the resistance drops in any path from any point $A$ to any point $B$, and $\Sigma e$ represents the *algebraic* sum of the electromotive forces in *this same* path, a current being considered as negative if its actual direction through the resistance $r$ is from $B$ toward $A$, and an electromotive force being considered as negative if its actual direction is from $B$ to $A$

The second relation may also be written

$$v_{BA} = \Sigma e - \Sigma ri \tag{2a}$$

where $v_{BA}$ represents the drop of potential from $B$ to $A$, or, what amounts to the same thing, the *rise* of potential from $A$ to $B$, the positive sense of both $e$ and $i$ as before being in the direction from $A$ to $B$.

It will avoid much confusion if the symbol $v_{AB}$ is always read "drop of potential from $A$ to $B$," and the symbol $v_{BA}$ is always read "rise of potential from $A$ to $B$," and similarly for any other pair of subscripts.

Kirchhoff's Second Law applied to a *closed loop* in a conducting network *i.e.*, to a path whose two ends $A$ and $B$ coincide, becomes $0 = \Sigma e - \Sigma ri$, since no difference of potential can exist between two points which coincide. Hence for any closed loop in a network

$$\Sigma e = \Sigma ri \tag{2b}$$

where $\Sigma e$ represents the *algebraic* sum of the electromotive forces acting around this loop, and $\Sigma ri$ represents the *algebraic* sum of the resistance drops in this loop. When applying this loop equation, the positive sense of each electromotive force and current must be taken in the same direction around the loop. It will avoid confusion to take always the clockwise direction as the positive sense. An electromotive force or current in the counter-clockwise direction is then to be considered as a negative quantity.

Equations (1) and (2b) enable one to write down a set of simultaneous equations for any given network, but it will be found that at least one of the current equations may be derived directly from the other current equations, and that at least one of the potential equations may be derived from the other potential equations. That is, the number of independent equations of each form will be at least one less than the number which it is possible to write down.

It should be particularly noted that it is frequently unnecessary to write down formally all the possible independent equations, for many of the simpler problems can be solved by writing down two independent expressions for the potential drop between each pair of points and equating these two expressions. Also, it is frequently unnecessary to use separate symbols for all the currents, but from an inspection of the diagram the currents in certain branches may be written down immediately as the sum or difference of the currents in other branches.

**49. Resistances and Electromotive Forces in Series.**—From Kirchhoff's Second Law it follows that when two or more conductors are connected in series, so that the same current flows through each, these several conductors are equivalent to a single conductor having a resistance equal to the arithmetical sum of the resistances of the several conductors, namely, to a resistance

$$r = r_1 + r_2 + r_3 + \text{etc.} \qquad (3)$$

where $r_1$, $r_2$, $r_1$, etc., are the resistances of the individual conductors. The heat dissipated in the single resistance $r$ by a current $I$ is also equal to the total heat dissipated in by this same current in the resistances $r_1$, $r_2$, $r_3$, etc.; see equation (4), Chapter II.

It also follows from Kirchhoff's Second Law that when two or more devices, through each of which the current has but a single

path, are connected in series, so that the same current flows through each, these several devices are equivalent to a single device having a resistance equal to the sum of the resistances of the several devices (see equation (3)) and an electromotive force equal to the **algebraic** sum of the electromotive forces of the several devices, namely, an electromotive force

$$E = E_1 + E_2 + E_3 + \text{etc.} \tag{4}$$

where $E_1$, $E_2$, $E_3$, etc. are the electromotive forces of the separate devices. In this expression all electromotive forces having the same direction as the current are to be represented by positive numbers, and those electromotive forces which have a direction opposite to that of the current are to be represented by negative numbers.

When the sum represented by equation (4) is a positive number, the several devices are equivalent to a single source of electric energy having an electromotive force numerically equal to this sum. Where this sum is a negative number, the several devices are equivalent to a single receiver of electric energy having a *back* electromotive force numerically equal to this sum.

A battery of two or more voltaic cells in series is a common illustration of the relations just stated. When each of $N$ identical cells are connected in series so that the positive terminal of each is connected to the negative terminal of the next in the series (*e.g.*, carbon to zinc, in the case of dry cells), the total electromotive force of the battery is $NE$, where $E$ is the electromotive force of each cell. Similarly the total internal resistance of the battery is $Nr + r_e$, where $r$ is the internal resistance of each cell, and $r_e$ is the total resistance of the connectors between the several cells. Should one cell in the chain be reversed, then its electromotive force would balance that of one of the other cells, and the resultant electromotive force would be $(N - 2)E$.

Problem 1.—Six dry cells are connected in series. Five of these cells are old ones, and each one has an electromotive force of 1.3 volts, and an internal resistance of 0.5 ohm. The sixth cell is a new one, having an electromotive force of 1.5 volts and an internal resistance of 0.06 ohm. By mistake, when this new cell is connected in series with the others, its carbon terminal is connected to the carbon terminal of one of the other cells. The combined resistance of the connectors between the several cells, including the "contact" resistance at the binding-posts, is 0.01 ohm.

(a) What is the total electromotive force of this battery as connected?
(b) What would be the electromotive force of this battery were the sixth

cell properly connected? (*c*) What is the total internal resistance of this battery as connected? (*d*) When the sixth cell is properly connected? (*e*) What is the maximum current which can be obtained from this battery as connected? (*f*) When the sixth cell is properly connected? (*g*) When the battery as connected supplies 1.5 amperes to an external circuit connected to its terminals, what will be the terminal voltage, assuming the electromotive forces of the cells to remain unaltered? (*h*) What must be the resistance of this external circuit?

*Answer.*—(*a*) 5.0 volts. (*b*) 8.0 volts. (*c*) 2.57 ohms. (*d*) 2.57 ohms. (*e*) 1.95 amperes. (*f*) 3.11 amperes. (*g*) 1.15 volts. (*h*) 0.76 ohm.

**Problem 2.**—An electric motor consists essentially of a stationary member, called the field structure, and a rotating member, called the armature. There is a winding of a number of turns of wire on each of these two parts. In a "series" motor the armature and field winding are connected in series. Due to the electromagnetic action between the two windings, an electromotive force is induced in the armature winding when the armature rotates (compare with the shunt dynamo described in Problem 17 of Chapter II).
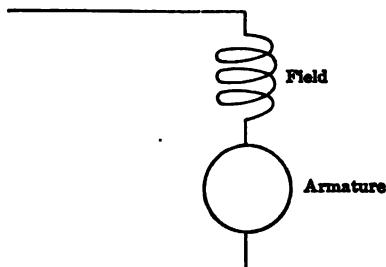


FIG. 27.—Series motor.

The total mechanical work done by the armature is equal to the product of this electromotive force by the current through the armature. A series motor is usually represented diagrammatically as shown in Fig. 27.

Motors used for traction work are almost invariably of the series type. The ordinary trolley car motor is designed to operate normally when from 500 to 600 volts are impressed across its terminals, this being the usual voltage maintained between the trolley wire and the track. In order to limit, to a safe value, the current taken by the motor at starting, a resistance is inserted in series with it, and as the armature speeds up this resistance is cut out of circuit.

(*a*) The field winding of a given 500-volt series motor has a resistance of 0.1 ohm and the armature winding a resistance of 0.2 ohm. When the two terminals of this motor are connected respectively to the trolley wire and track, how much resistance must be inserted in series with it, in order to limit the current, when the armature is at rest, to 300 amperes? (*b*) When this motor is developing a given amount of mechanical power, with the starting resistance completely cut out (short-circuited), the motor takes a current of 100 amperes. What is the value of the electromotive force developed in

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its armature? (c) How much mechanical power is developed? (d) What is the voltage drop through the field winding? (e) Through the armature? (f) If two such motors are connected in series and each develops the same amount of mechanical power, what will be the electromotive force of each when the current is 100 amperes and the voltage between the trolley and track is 500 volts?

*Answer.*—(a) 1.37 ohms. (b) 470 volts. (c) 47 kilowatts or 63.0 horse-power. (d) 10 volts. (e) 20 volts. (f) 220 volts.

**50. Voltage Drop and Power Loss in a Transmission Line Supplying a Single Load.**—A problem of frequent occurrence in engineering work is to determine the difference between the terminal voltage of a generator (or the voltage at the switchboard) and the voltage which exists between the terminals of a given load (such as lamps or motors) to which the generator supplies power, and also what will be the power lost (*i.e.*, dissipated as heat) in the wires forming the transmission line (see Fig. 28).
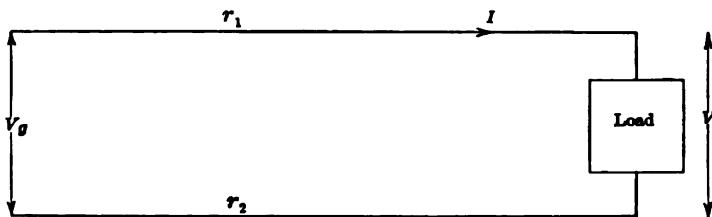


FIG. 28.

Let $V$ = voltage between the terminals of the load.

$V_g$ = voltage between the generator terminals.

$P$ = the power, in watts, taken by the load.

$r = r_1 + r_2$ = the total resistance, in ohms, of the transmission line, where $r_1$ and $r_2$ are the resistances respectively of the two wires which form the transmission line.

Then the total current through the load and line will be (see Article 40)

$$I = \frac{P}{V} \tag{5}$$

From Ohm's Law (equation (6b), Chapter II), the drop of potential from the positive terminal of the generator to the positive terminal of the load will be $r_1 I$, and similarly the drop of potential from the negative terminal of the load to the negative terminal of the generator will be $r_2 I$. Hence, the difference

between the terminal voltage of the generator and the voltage impressed on the load is

$$V_g - V = (r_1 + r_2) I = \frac{rP}{V} \qquad (6)$$

The *per cent.* ratio of the voltage drop in the line to the voltage across the terminals of the load is then

$$100 \frac{V_g - V}{V} = \frac{100 \, rP}{V^2} \qquad (7)$$

From Joule's Law (equation (4) of Chapter II), the total power loss in the two wires is

$$P_L = rI^2 = \frac{rP^2}{V^2} \qquad (8)$$

The *per cent.* ratio of the power lost to the power delivered to the load is

$$100 \frac{P_L}{P} = \frac{100 \, rP}{V^2} \qquad (9)$$

Note that for a constant amount of power $P$, delivered over a line of given resistance, both the per cent. voltage drop and the per cent. power loss vary inversely as the square of the voltage impressed on the load. In other words, doubling the impressed voltage, for a line of given resistance, reduces the loss in the line, for a given amount of power delivered, to one-fourth of its original value.

Again, since the resistance of a conductor of given length and of given material (*e.g.*, copper) is inversely proportional to the total weight of the conductor (see equation (15), next article), it follows that for a given amount of power delivered with a given allowable loss in the line, the weight of conductor required is inversely proportional to the square of the voltage impressed on the load.

The direct-current line voltages ordinarily employed in practice are 100 to 110 volts for incandescent lamps and small motors, 200 to 250 volts for larger motors, and 500 to 600 volts for ordinary railway motors. Direct-current voltages up to 5000 volts, however, have of late been successfully employed in railway work. Higher voltages than those stated would of course result in a saving in the weight, and therefore in the cost, of the conductors required for the transmission line, but safety and economy in the

construction of the apparatus constituting the load in general more than offset any saving which would result in using voltages higher than the values stated (see, however, Article 53).

It should also be noted, from equations (7) and (9), that for a given load and a given transmission line, the per cent. power loss and the per cent. voltage drop are equal. This relation, however, does not hold for alternating currents; see Chapter XV.

**Problem 3.**—A 250-volt, 50-horsepower motor in a factory is supplied with electric power from a generator which is 400 feet away. Each wire of the transmission line connecting the generator and motor has a resistance of 0.06 ohm. The motor has an efficiency of 90 per cent. and develops 50 mechanical horsepower when a pressure of 220 volts is maintained across its terminals.

(a) What is the current in each line wire? (b) By how many volts does the generator voltage exceed the voltage impressed on the motor? (c) What is the per cent. voltage drop in the line? (d) What is the per cent. power loss in the line? (e) What is the total generator output under these conditions? (f) What is the efficiency of transmission (i.e., percentage ratio of the input to the load to the output of the generator)?

*Answer.*—(a) 188 amperes. (b) 22.6 volts. (c) 10.27 per cent. (d) 10.27 per cent. (c) 45.7 kilowatts. (f) 90.7 per cent.

**Problem 4.**—A motor is to be supplied with power from a generator which is 300 feet away. The maximum power developed by the motor is 25 horsepower, and at this load the efficiency of the motor is 92 per cent. The voltage at the generator terminals is 120. What must be the resistance per 1000 feet of each wire of the transmission line, in order that the power lost in the line shall not exceed 10 per cent. of the power delivered to the load?

*Answer.*—0.0977 ohm. per 1000 feet (see also Problem 10 of Article 61).

**Problem 5.**—Using the notation given at the beginning of this article:

(a) Prove that for a given voltage $V_g$ at the generator the voltage at the load is

$$V = \frac{1}{2} V_g \left[ 1 \pm \sqrt{1 - \frac{4rP}{V_g^2}} \right]$$

(b) What is the significance of the double sign before the radical? (c) What is the maximum power which can be transmitted over a line of resistance $r$ when the impressed voltage at the sending (or generator) end is $V_g$? (d) What is the ratio of the power lost in the line to the power delivered to the load under these conditions?

*Answer.*—(b) For a given amount of power delivered there are two possible load voltages, one greater than half the generator voltage and one less than half this voltage. In the first case the efficiency of transmission is greater, and in the second case less, than 50 per cent. (c) $P_m = \frac{V_g^2}{4r}$. (d) The power lost in the line is equal to the power delivered to the load.

**51. Resistances and Electromotive Forces in Parallel.**—From Kirchhoff's two laws it follows that when any number of resistances $r_1$, $r_2$, $r_3$, etc., are connected in parallel between two points $A$ and $B$ (Fig. 29), and there are electromotive forces $E_1$, $E_2$, $E_3$, etc. in these respective branches, the drop of potential $V_{AB}$ from $A$ to $B$ must be the same for each branch, and the total current $I$ entering the junction $A$ (or leaving the junction $B$) must be equal to the algebraic sum of the currents $I_1$, $I_2$, $I_3$, etc. in the several branches.

The mathematical expression of these relations is that

$$I = I_1 + I_2 + I_3 + \text{etc.} \tag{10}$$
$$V_{AB} = r_1 I_1 - E_1 \tag{11}$$
$$= r_2 I_2 - E_2 \tag{11a}$$
$$= r_3 I_3 - E_3 \tag{11b}$$
$$= \text{etc.}$$
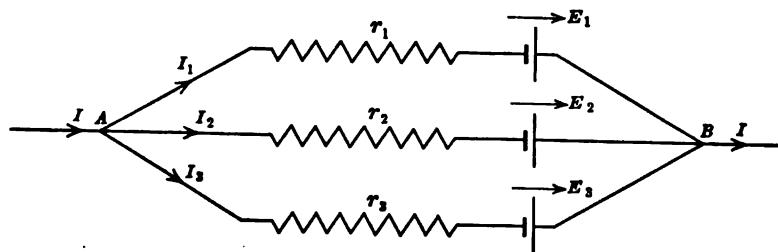


FIG. 29.

Note that in these expressions the positive sense of the potential drop, and of each current and electromotive force in the section of the circuit under consideration, are taken as the direction from $A$ to $B$. From these equations the unknown quantities may be calculated when the number of *independent* equations do not exceed in number the number of branches.

In general, it is not possible to consider two or more parallel circuits equivalent to a single circuit. In the special case, however, when the *electromotive forces in the several branches are equal and in the same direction* (with respect to the potential drop from $A$ to $B$), the several branches may be considered as equivalent to a single circuit having a resistance $r$, where

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \text{etc.} \tag{12}$$

and an electromotive force equal to that of each branch.

This relation follows immediately from the equations just given when $E_1$, $E_2$, $E_3$, etc., are put equal to $E$, for then

$$I_1 = \frac{1}{r_1} (V_{AB} + E)$$

$$I_2 = \frac{1}{r_2} (V_{AB} + E)$$

$$I_3 = \frac{1}{r_3} (V_{AB} + E)$$

etc.  etc.

Whence

$$I = I_1 + I_2 + I_3 + \text{etc.}$$
$$= \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) (V_{AB} + E)$$

which is identical with the relation which would hold for a single circuit having the resistance $r$, as given by equation (12), and, an electromotive force $E$.

In terms of the total current, the currents in the several branches may then be written

$$I_1 = \frac{r}{r_1} I, \qquad I_2 = \frac{r}{r_2} I, \qquad I_3 = \frac{r}{r_3} I, \text{etc.} \qquad (13)$$

That is, when the electromotive forces in the several branches are equal and in the same direction, or when there is no electromotive force in any one of the branches, the total current divides among the several branches inversely as their resistances.

Also note that when the electromotive forces in the several branches are equal, the total power dissipated as heat is

$$P_h = r_1 I_1{}^2 + r_2 I_2{}^2 + r_3 I_3{}^2 + \text{etc.}$$
$$= \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) (V_{AB} + E)^2 = rI^2$$

which is the same as would be dissipated in a single conductor having the resistance $r$ (equation (12), and carrying a current $I$.

In the special case of but two branches in parallel, and each containing equal electromotive forces in the same direction, or when there is no electromotive force in either branch, equation (12) for the equivalent resistance may be written

$$r = \frac{r_1 r_2}{r_1 + r_2} \qquad (14)$$

An extremely useful deduction from equations (3) and (12) is that the resistance of a straight wire of a given material and of uniform cross-section throughout its length is directly proportional to its length and inversely proportional to its cross-section.

Let $l$ be the length of the given wire $W$ and let $S$ be its cross-section, and let $l_1$ be the length and $S_1$ the cross-section of any other wire $W_1$ of the same material. The wire $W$ may then be considered as made up of $\frac{lS}{l_1S_1}$ wires identical with $W_1$, so arranged that they form $\frac{S}{S_1}$ parallel paths and each of these paths may be considered as made up of $\frac{l}{l_1}$ wires in series, each of the same length and cross-section as $W_1$.

Let $r_1$ be the resistance of the wire $W_1$. Then the resistance of each of these parallel paths is $\frac{l}{l_1}r_1$, and the resistance of $\frac{S}{S_1}$ such paths in parallel, is

$$r = \frac{lS_1}{l_1S}r_1$$

Whence

$$\frac{r}{r_1} = \frac{l}{l_1}\cdot\frac{S_1}{S} \tag{15}$$

That is, the resistance of the wire $W$ is to the resistance of the wire $W_1$ directly as the length of these two wires and inversely as their cross-sections.

The argument just given also leads to the conclusion that the current through any area $S_1$ of the total cross-section $S$ of a given straight wire which has a uniform cross-section throughout its length, is directly proportional to the area $S_1$ (see equation (13)). That is, the current density in a straight wire, of uniform cross-section throughout its length, has the same value at every point of the wire, provided the electromotive forces, if any, in the various possible parallel paths through the wire are equal. In other words, under the conditions specified, the current density is uniform throughout the wire; see however Article 54.

The above deductions in regard to the variation of the resistance of a wire with its length and cross-section and in regard to the current density in the wire, also hold for a wire, rod or bar bent into a curve (*e.g.*, wound into a coil), provided the radius of curvature of this curve is large compared with the diameter

of the conductor, or with that dimension of the conductor which is in line with this radius of curvature.

The variation of the resistance of a conductor with the dimensions, material, and shape of the conductor is treated in greater detail in Chapter İV, as is also the effect of temperature on resistance.

When unequal electromotive forces exist in the several parallel branches between any two points $A$ and $B$ (*e.g.*, when one of two parallel branches contains an electromotive force and the other does not), equations (12) and (13) do not hold. For example, consider two branches, each containing a source of electromotive
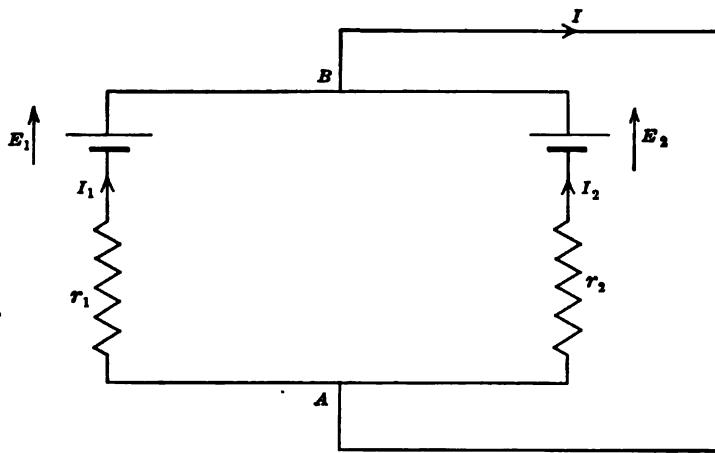


Fig. 30.

force, as shown in Fig. 30. This arrangement is quite common in practice; *e.g.*, two batteries in parallel or any two electric generators in parallel, supplying a total current $I$ to some external load. When this total current $I$ and the electromotive forces $E_1$ and $E_2$ and the resistances $r_1$ and $r_2$ of the two branches are known, the currents $I_1$ and $I_2$ in two branches and the terminal voltage $V_{AB}$ of each may be found from the three equations

$$I = I_1 + I_2$$
$$E_1 - r_1I_1 = E_2 - r_2I_2$$
$$V_{BA} = E_1 - r_1I_1$$

The solution of these three equations is

$$I_1 = \frac{r_2}{r_1 + r_2} I + \frac{E_1 - E_2}{r_1 + r_2}$$

$$I_2 = \frac{r_1}{r_1 + r_2} I - \frac{E_1 - E_2}{r_1 + r_2}$$

$$V_{BA} = \left[ E_1 - r_1 \left( \frac{E_1 - E_2}{r_1 + r_2} \right) \right] - \left( \frac{r_1 r_2}{r_1 + r_2} \right) I$$

The total power dissipated as heat in the two branches is then

$$P_h = r_1 I_1{}^2 + r_2 I_2{}^2$$

$$= \left( \frac{r_1 r_2}{r_1 + r_2} \right) I^2 + (r_1 + r_2) \left( \frac{E_1 - E_2}{r_1 + r_2} \right)^2$$

An inspection of Fig. 26 shows that the term $\dfrac{E_1 - E_2}{r_1 + r_2}$, which enters into these equations, is equal to the current, in the direction of $E_1$, which would exist in the closed loop formed by the two parallel branches were there no current supplied to the load, *i.e.*, were $I$ equal to zero. This current

$$I_c = \frac{E_1 - E_2}{r_1 + r_2} \tag{16}$$

may be called the "circulatory current" between the two branches. This circulatory current represents a transfer of energy from one branch of the circuit to the other, but no transfer of energy to the load.

Put

$$r = \frac{r_1 r_2}{r_1 + r_2} \tag{17}$$

Then, from the relations just deduced, the currents in the two branches are respectively

$$I_1 = \frac{r}{r_1} I + I_c \tag{18}$$

$$I_2 = \frac{r}{r_2} I - I_c \tag{18a}$$

The *rise* of potential from $A$ to $B$ is

$$V_{BA} = E_1 - rI - r_1 I_c \tag{19}$$

and the total power dissipated as heat in the two branches is

$$P_h = rI^2 + (r_1 + r_2)I_c{}^2 \tag{20}$$

Comparing these relations with those which would hold were the electromotive forces in the two branches equal and in the same

direction, it is seen that, due to the inequality in the two electromotive forces:

(a) The current in one branch is increased and the current in the other decreased.

(b) The terminal voltage is decreased by an amount equal to the resistance drop of the circulatory current in the branch containing the larger electromotive force.

(c) The power dissipated as heat is *always* more than it would be were the electromotive forces in the two branches equal.

**Problem 6.**—Three identical coils are first connected in series and then in parallel. What is the relative resistance of the series and parallel arrangement?

*Answer.*—The coils in series will have 9 times the resistance of the coils in parallel.

**Problem 7.**—A copper wire 1 foot long and 0.001 inch in diameter has a resistance at 20°C. of 10.4 ohms. What will be the resistance, at this same temperature, of a coil of copper wire having 200 turns, if the diameter of the wire is 0.1 inch and the mean length of each turn in the coil is 22 inches (mean diameter 7 inches)?

*Answer.*—0.381 ohm.

**Problem 8.**—When voltage is impressed, from some external source, across the terminals of a shunt dynamo (see Problem 17 of Article 40), the machine will run as a motor, and its armature will do mechanical work. The armature, of course, does mechanical work only when it is actually rotating. In a certain shunt motor the resistance of the field winding is 20 ohms and the resistance of its armature is 0.02 ohm. The maximum current which can flow through the armature without damaging it is 1000 amperes.

(a) If the armature is at rest and 220 volts are impressed across its terminals, what will be the current through the field winding and through the armature winding? (b) How may the armature current at starting be limited to a safe value? (c) How much resistance would have to be placed in series with the armature at starting to limit the armature current to 1000 amperes? (d) When the motor is developing a given amount of mechanical power, the total current taken by it when 220 volts are impressed on its terminals, is 300 amperes, the starting resistance all having been cut out. What is the current through its armature? (e) What is the back electromotive force developed in its armature under these conditions? (f) How much mechanical power is developed by the motor under these conditions?

*Answer.*—(a) 11 amperes through the field and 11,000 amperes through the armature. (b) By inserting a resistance in series with the armature. (c) 0.2 ohm. (d) 289 amperes. (e) 214.22 volts. (f) 83.2 horsepower.

**Problem 9.**—Two batteries which have internal electromotive forces of 48 and 50 volts, and internal resistances of 0.025 and 0.075 ohms respectively, are connected in parallel and together supply a current of 100 amperes.

(a) What is the terminal voltage of each battery? (b) How much current is supplied by each? (c) How much power is developed as heat in the two batteries?

*Answer.*—(a) 46.625 volts. (b) The battery which has an electromotive force of 50 volts supplies 45 amperes and the other 55 amperes. (c) 227.5 watts.

**Problem 10.**—If the electromotive force of the second battery in Problem 9 were equal to that of the first battery, namely, 48 volts:

(a) What would be the terminal voltage of each battery? (b) How much current would be supplied by each? (c) How much power would be dissipated as heat in the two batteries?

*Answer.*—(a) 46.125 volts. (b) 75 amperes from the battery having the lower resistance and 25 amperes from the other. (c) 187.5 watts.

**Problem 11.**—In Fig. 31, *A* and *B* represent two substations which together supply electric power to such cars as may be on the section of track between them. The trolley wire at each substation is connected to the positive busbar, and the two track rails (in parallel) at each substation are



Trolley Wire

Car

Track
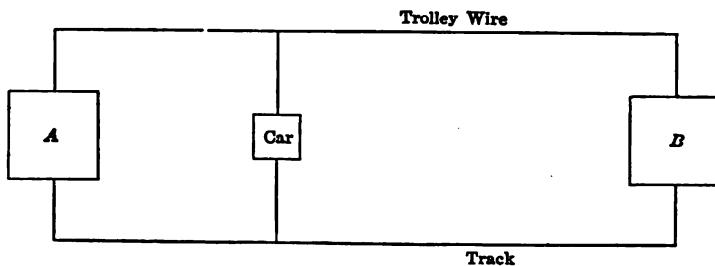
FIG. 31.

connected to the negative busbar. The voltage between the positive and negative busbars at *A* is 600, and the busbar voltage at *B* is 605. The distance between *A* and *B* is 5 miles. At a given instant there is but one car between *A* and *B*, and the distance of this car from *B* is twice its distance from *A*. The trolley wire has a resistance of 0.05 ohm per 1000 feet, and each track-rail has a resistance of 0.02 ohm per 1000 feet. At the given instant the car is taking a current of 150 amperes.

(a) How much current is supplied by each substation? (b) What is the voltage at the car between the trolley and the track?

*Answer.*—(a) 96.8 amperes from *A* and 53.2 amperes from *B*. (b) 548.9 volts.

**52. Single Transmission Line Supplying Two or More Loads.**— It frequently happens that it is both convenient and economical to connect several separate loads to a single feeder from a power-house or substation, as shown diagrammatically in Fig. 32. When the currents taken by these several loads, the resistances of the

various sections of the feeder, and the voltage at the far end of the feeder are known, then the voltage across the various loads, the total current taken by the feeder, and the busbar voltage may be readily calculated. As in the preceding circuit calculations, the only fundamental principles involved are Kirchhoff's Laws.

For example, referring to Fig. 32, let $V_B$ be the voltage across the load $B$, $I_B$ the current taken by this load, $I_A$ the current taken by the load $A$, and $r_1$ and $r_2$ respectively the resistances of the feeder between the power-house and load $A$ and between load $A$ and load $B$. $r_1$ and $r_2$ represent the resistances of both of the wires which form the respective sections of the feeder.

Then the current in the section of the feeder between $A$ and $B$ is $I_B$, and the current in the section of the feeder between the power-house and $A$ is
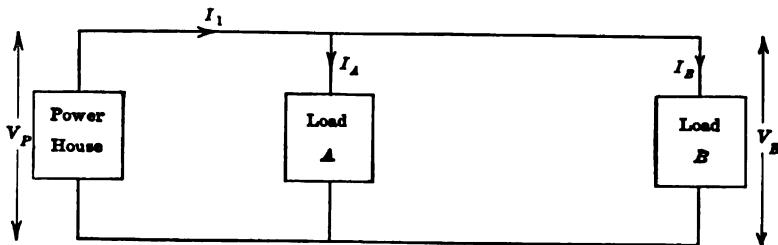


FIG. 32.

$$I_1 = I_A + I_B$$

The voltage at the load $A$ is then

$$V_A = V_B + r_2 I_B$$

and the voltage at the power-house busbars is

$$V_P = V_A + r_1 I_1$$

or

$$V_P = V_B + r_1(I_A + I_B) + r_2 I_B$$

The total power lost in the line is

$$P_h = r_1 I_1^2 + r_2 I_B^2$$

or

$$P_h = r_1(I_A + I_B)^2 + r_2 I_B^2.$$

In a similar manner, the required generator voltage, the total power lost in the line, and the voltages at the various loads, may

be calculated for any number of loads on the feeder, provided the resistances of the various sections of the feeder and the voltage at the far end of the line are known.

**Problem 12.**—The trolley cars on a certain section of track are supplied with power from a single substation, the trolley wire being connected to the positive busbar of the substation, and the two rails in parallel being connected to the negative busbar of this substation. At a given instant there are two cars on this section of track. One car is at a distance of 0.5 mile from the substation and takes a current of 100 amperes. The other car is at a distance of 2 miles from the substation and takes a current of 50 amperes. At the further car the voltage between the trolley and track is 500. The resistance of the trolley wire per 1000 feet is 0.05 ohm, and the resistance of each rail per 1000 feet is 0.02 ohm.

(a) What is the voltage between the trolley and track at the car nearer the substation? (b) What is the busbar voltage at the substation? (c) What is the total power lost in the trolley and track? (d) What is the difference between the potential of a point on the rail directly opposite the power-house and a point on the rail directly under the further car? (e) What is the direction of the drop of potential in the rail? (f) What is the ratio of the total voltage drop in the track to the total voltage drop in the trolley wire?

*Answer.*—(a) 523.8 volts. (b) 547.5 volts. (c) 4.75 kilowatts. (d) 7.92 volts. (e) The drop is from the far end of the line *toward* the substation. (f) 20 per cent.

**53. Three-wire System of Distribution.**—As noted in Article 50, the weight, and therefore the cost, of the conductors required to transmit a given amount of power with a given efficiency of transmission (or with a given difference between the voltages at the sending and receiving ends of the line) is inversely proportional to the square of the voltage maintained between the two wires of the line. On the other hand, experience has shown that the most satisfactory voltage for incandescent lamps for interior use is from 100 to 120. To secure the advantage of a higher transmission voltage, and at the same time keep the required lower voltage on the lamps, the three-wire system of distribution, as shown schematically in Fig. 33, is commonly used.

At the generator end a voltage slightly greater than the required lamp voltage is maintained between *each* of the outside wires and the middle, or neutral, wire. Various methods for doing this are in use, the simplest (though seldom used) method is to connect two generators in series, and to connect the neutral wire to the junction between the two generators. The voltage impressed across each lamp load is then maintained equal, to a

close approximation, to the rated voltage of the lamps, for example, 110 volts. The same line may also be used to supply 220-volt motors, by connecting the motors, as shown, between the two outer wires, thereby making possible the use of more economical motors.

Were the lamp loads on the two sides of the system connected to the neutral wire at the same point, then the current carried by this wire would be only the *difference* between the currents taken by the two lamp loads. The maximum value of this current would then never exceed the current taken by the larger of the two lamp loads, and therefore the neutral wire need not be larger than each outside wire. When the various lamp loads are separated, as shown in the figure, the currents in the various sections of the neutral wire will have different values (and may also have different directions), but at no point will this current exceed in value the larger of the currents in the two outer wires where the
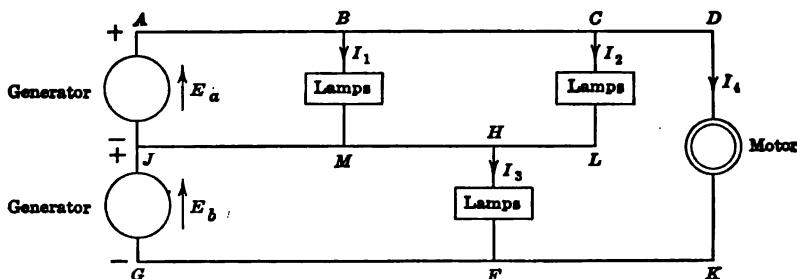


FIG. 33.—Three-wire system

latter are connected to the generators. In practice, therefore, the three wires are usually all of the same cross-section.

When the currents taken by the several loads, the electromotive forces of the generators, the resistances of the various sections of the wires, and the internal resistances of the generators are known, then the current in each section of wire, the voltage across each load, and the terminal voltage of each generator may be calculated by applying Kirchhoff's Laws. Such calculations may be greatly simplified by using the fewest symbols possible.

For example, referring to Fig. 33, instead of introducing separate symbols for the currents in the several sections of the wires, note that the current from $C$ to $D$ and from $K$ to $F$ is $I_4$, the current from $B$ to $C$ is $(I_2 + I_4)$, the current from $L$ to $H$ is $I_3$,

the current from $H$ to $M$ is $(I_2 - I_3)$, the current from $F$ to $G$ is $(I_3 + I_4)$, the current in the generator from $G$ to $J$ is also $(I_3 + I_4)$, the current from $M$ to $J$ is $(I_1 + I_2 - I_3)$, the current in the generator from $J$ to $A$ is $(I_1 + I_2 + I_4)$, and the current from $A$ to $B$ is also $(I_1 + I_2 + I_4)$.

It should also be noted that any change in the lamp load on one side of the system will change only slightly the voltage across the lamps on the other side, provided the neutral wire is connected to the junction between the two generators (or other source of power). Were the neutral not so connected, then the two sides of the system would be in series and would therefore take the same current. As a consequence, the voltages across the loads on the two sides of the system would be directly proportional to the equivalent resistances of these loads. For example, were the voltage between the outer wires 220, and were 5 lamps connected to one outer wire and 100 equal lamps (in parallel) connected to the other outer wire, the voltage across the 5 lamps would be $\frac{100}{105} \times 220 = 209.5$ and the voltage across the 100 lamps would be $\frac{5}{105} \times 220 = 10.5$. The result would be that the 5 lamps would burn out, which in turn would interrupt the current through the other lamps.

**Problem 13.**—Two loads of equal resistance are connected in series and are supplied with power over a two-wire transmission line. The voltage across each load is to be $V$ volts, and each load takes $P$ watts. The permissible power loss in the transmission line is $q$ per cent. of the total power delivered to the two loads.

(a) Prove that the resistance of each line wire must be $r_1 = \frac{qV^2}{100P}$. (b) Prove that, were the two loads connected in parallel and supplied at the voltage $V$ over a two-wire line, with the same total power loss as before, the resistance of each line wire would have to be $r_2 = \frac{qV^2}{400P}$. (c) What would be the relative total weights of wire in these two cases? (d) If the two loads were supplied over a three-wire system, with a neutral wire of the same size as each outer, what would be the total weight of wire required, as compared with the total weight of conductor required to transmit the same total amount of power, at the same efficiency, over a two-wire system with the same voltage between wires as between each outer and the neutral in the three-wire system?

*Answer.*—(c) The weight of wire required with the loads in series is only 25 per cent. of that required with the loads in parallel. (d) 37.5 per cent.

**Problem 14.**—In a three-wire system in which the loads are distributed as shown in Fig. 33, the currents taken by the loads are:

$I_1 = 20$ amperes, $\qquad$ $I_3 = 40$ amperes,
$I_2 = 50$ amperes, $\qquad$ $I_4 = 30$ amperes.

The resistances of the various sections of the wires are:

$AB = 0.05$ ohm, $\qquad$ $FK = 0.09$ ohm,
$BC = 0.12$ ohm, $\qquad$ $JM = 0.05$ ohm,
$CD = 0.03$ ohm, $\qquad$ $MH = 0.04$ ohm,
$GF = 0.11$ ohm, $\qquad$ $HL = 0.08$ ohm.

The internal resistance of each generator is 0.1 ohm. The terminal voltage of each generator is 120 volts.

(a) What is the value and direction of the current in each section of the wires? (b) What is the voltage across each load? (c) What is the current supplied by each generator? (d) What must be the electromotive force of each generator? (e) What is the total power dissipated as heat in the line wires?

*Answer.*—100 amperes from $A$ to $B$; 80 amperes from $B$ to $C$; 30 amperes from $C$ to $D$ to $K$ to $F$; 70 amperes from $F$ to $G$; 50 amperes from $L$ to $H$; 10 amperes from $H$ to $M$; 30 amperes from $M$ to $J$. (b) 113.5 volts from $B$ to $M$; 99.5 volts from $C$ to $L$; 114.2 volts from $H$ to $F$; 214.1 volts from $D$ to $K$. (c) 100 amperes from generator $a$ and 70 amperes from generator $b$. (d) 130 volts for generator $a$ and 127 volts for generator $b$. (e) 2.164 kilowatts.

[NOTE: In practice, the transmission line wires are usually of sufficiently low resistance to keep the voltages across the several lamp loads equal to within 2 or 3 per cent.]

**Problem 15.**—If the equivalent resistances of the four loads in Fig. 33 were respectively:

$r_1 = 4.92$ ohms, $\qquad$ $r_3 = 10.1$ ohms,
$r_2 = 11.7$ ohms, $\qquad$ $r_4 = 7$ ohms,

and the other resistances as given in Problem 14, and the terminal voltages of the generators each 120 volts:[1]

(a) What would be the voltage across each load? (b) How much power would be supplied to the transmission line by each generator?

*Answer.*—(a) 115.7 volts from $B$ to $M$; 110.1 volts from $C$ to $L$; 116.2 volts from $H$ to $F$; 223.2 volts from $D$ to $K$. (b) 7.78 kilowatts from generator $a$ and 5.21 kilowatts from generator $b$.

[1]Problems of this kind are most readily solved by the use of determinants; see the article on "Equations" in PENDER'S HANDBOOK FOR ELECTRICAL ENGINEERS.

## RESISTANCE AND CONDUCTANCE

### THEIR DEPENDENCE ON THE MATERIAL, DIMENSIONS, SHAPE AND TEMPERATURE OF THE CONDUCTOR

**54. General.**—As noted in Article 51, the resistance of a straight wire of uniform cross-section is proportional to its length and inversely proportional to its cross-section, and also depends upon the nature of the material of which it is made and its temperature. The proportionality between the resistance and the length of a conductor, and the inverse proportionality between the resistance and the cross-section of a conductor, hold only when (1) the two end surfaces of the conductor are parallel, (2) the cross-section of the conductor is of the same shape and dimensions throughout, and (3) the electromotive force, if any, within the conductor has the same value in every path through it. Under these conditions the flow of electricity through the conductor is along lines parallel to its axis, each of these lines of flow has the same length, and the intensity of the current through each unit area in the cross-section of the conductor has the same value.

When any one of the three conditions stated is not fulfilled, the flow of electricity through the conductor may not be along parallel lines, and the quantity of electricity which flows through equal areas in any cross-section will in general be different. The lines of flow may be divergent or convergent, either straight or curved, and either of the same or of unequal lengths. The particular shape and distribution of these lines of flow, or "stream lines," depend upon both the shape of the conductor and upon the location of the terminals at which the current enters and leaves it.

**55. Resistivity or Specific Resistance.**—Consider first the case of a wire or bar of length $l$ and of the same cross-section throughout its length, and with parallel end surfaces. Then, as shown in Article 51, when all parts of each end-surface are respectively at the same electric potential, and when there is no electromo-

tive force in the wire, the resistance $r$ of the wire is directly proportional to its length $l$ and inversely proportional to its cross-sectional area $S$, namely,

$$r = \rho \frac{l}{S} \qquad (1)$$

where $\rho$ is a factor of proportionality. This factor $\rho$ is called the "resistivity," or "specific resistance," of the wire.

The resistivity of a conductor depends (1) upon the material of the conductor and (2) upon its temperature. The numerical value of the resistivity, namely, the numerical value of the factor $\rho$ in equation (1), also depends upon the units in which the resistance $r$, the length $l$ and the cross-section $S$ are expressd.

From equation (1) it follows that the resistivity of a conductor is numerically equal to the resistance of a wire, or bar, which has unit length and unit cross-sectional area, provided the stream lines of the current through this bar are all parallel to its axis, and the end-surfaces of the bar are parallel "equipotential" surfaces.

In particular, when the centimeter is used as the unit of length, the square centimeter as the unit of area, and the ohm as the unit of resistance, the resistivity is equal to the resistances in ohms of a cube of the material each edge of which is 1 centimeter long. The unit of resistivity when expressed in these units is therefore often called the "ohm per centimeter-cube." In order, however, to prevent the misconception that the resistance of a given mass of conductor is proportional to its volume, it is preferable to use the name "ohm-centimeter" for this unit. This signifies that the "dimensions" of resistivity are the same as the product of ohms by centimeters, *i.e.*,

$$\text{resistivity} = \frac{(\text{ohms}) \times (\text{centimeters})^2}{\text{centimeters}} = (\text{ohms}) \times (\text{centimeters})$$

When the c.g.s. system of units is employed, the resistivity of metals is usually expressed in microhm-centimeters, instead of ohm-centimeters, in order to avoid the use of small decimal fractions. For example, the resistivity of ordinary copper wire at 20°C. is 1.72 microhm-centimeters (= 0.00000172 ohm-centimeter). The resistivity of good insulators is expressed in megohm-centimeters; for example, the resistivity of glass is approximately $10^9$ megohm-centimeters (= $10^{15}$ ohm-centimeters).

Resistivity is also frequently expressed in ohm-inches (ohms per inch-cube), microhm-inches (microhms per inch-cube), or megohm-inches (megohms per inch-cube). Note that a resistivity of 1 ohm-centimeter is the same as a resistivity of $\dfrac{1}{2.54}$ ohm-inch. This follows from the fact that the numerical value of resistivity is proportional to the first power of the length of the edge of the unit cube, and not to the cube of this length.

It is customary in English-speaking countries to specify the length of a wire in feet and the cross-section in "circular mils." By a circular mil is meant the area of a circle which has a diameter of 0.001 inch, *i.e.*, a diameter of 1 "mil." Since the area of a circle varies as the square of its diameter in mils, this unit of area is therefore very convenient for expressing the area of the cross-section of a circular wire, for its use eliminates the factor $\pi$. For example, a circular wire having a diameter of 0.5 inch, has an area of $(500)^2 = 250,000$ circular mils.

When, in equation (1), $r$ is expressed in ohms, $l$ in feet, and $S$ in circular mils, the factor $\rho$ is the resistance in ohms of a wire 1 foot long and 1 circular mil in cross-section. This unit of resistivity is called the "ohm per mil-foot." Note that

$$1 \text{ ohm per mil-foot} = 0.1662 \text{ microhm-centimeter}$$
$$= 0.06545 \text{ microhm-inch.}$$

Equation (1) is directly applicable to the calculation of the resistance of a wire of uniform cross-section when its specific resistance is given, but care must be employed to express all the quantities entering into the formula in the *same system* of units. For example, the resistance in *ohms* of a wire which has a resistivity of 1.6 microhm-centimeter, a length of 1000 feet and a cross-section of $\frac{1}{4}$ square inch, is

$$r = 1.6 \times 10^{-6} \times \frac{1000 \times 12 \times 2.54}{0.25 \times (2.54)^2} = 0.0302 \text{ ohm.}$$

Or, since a resistivity of 1.6 microhm-centimeter is equivalent to a resistivity of $\dfrac{1.6}{2.54} = 0.630$ microhm-inch,

$$r = 0.630 \times 10^{-6} \times \frac{1000 \times 12}{0.25} = 0.0302 \text{ ohm.}$$

Or again, since a resistivity of 1.6 microhm-centimeter is equivalent to a resistivity of $\dfrac{1.6}{0.1662} = 9.63$ ohms per mil-foot, and

since 0.25 square inch $= \dfrac{4}{\pi} \times 0.25 \times 10^{6} = 318{,}000$ circular mils,

$$r = 9.63 \times \frac{1000}{318{,}000} = 0.0302 \text{ ohm.}$$

The resistance of a given length of wire of uniform cross-section is independent of the shape into which the wire is bent, provided the diameter of the wire is small compared to the radius of curvature of the curve into which it may be bent, *i.e.*, provided the stream lines of the current are all of the same length. This condition is almost always realized in practice, and consequently equation (1) is in general directly applicable to the calculation of the resistance of a wire, whether the wire be straight, or curved, or wound into a coil of any shape.

The three metals used commercially for conductors are copper, aluminum and iron (in the form of steel). The resistivity of copper at 20°C. is 1.7241 microhm-centimeters. (This is for so-called 100 per cent. conductivity copper; see Article 59.) The only metal which has a resistivity less than copper is silver, the resistivity of which is 1.59 microhm-centimeters at 20°C.

Aluminum has a resistivity about 60 per cent. greater than that of copper, but, due to its lower specific gravity, the weight of aluminum required for a transmission line of given length and given resistance is less than the weight of copper required for a line of the same length and resistance.

Steel wire has a resistivity about twelve times that of copper Steel rails have a resistivity from eight to twelve times that of copper, depending upon the chemical composition and previous heat treatment of the rails. For numerical values of the resistivity of various substances see any electrical engineers' handbook.

Any impurity in a metal increases its resistivity. All alloys have a greater resistance than that of the best conductor in them. For certain purposes, particularly in the construction of resistance boxes and rheostats, a high specific resistance is desirable. Various alloys are made for this purpose, the specific resistances of which can be found in any electrical engineers' handbook.

A conductor which is used on account of its high resistivity is sometimes called a "resistive conductor." A coil or grid, or an assembly of coils or grids, made of resistive conductors, is called

a "resistor." A resistor which is provided with means for varying the resistance between its terminals (*e.g.*, a contact dial and handle) is called a "rheostat." An assembly of small coils of known resistance arranged so that two or more of these coils may be connected in series or in parallel is called a "resistance box." The coils of a resistance box are designed to carry only a very small current, usually but a small fraction of an ampere.

**Problem 1.**—(*a*) What is the resistivity of copper in ohms per mil-foot at 20°C? (*b*) What is the cross-section in circular mils of a wire which has a diameter of 0.1 inch? (*c*) What is the resistance at 20°C. of 1000 feet of copper wire having a diameter of 0.1 inch?

*Answer.*—(*a*) 10.37 ohms per mil-foot. (*b*) 10,000 circular mils. (*c*) 1.037 ohms.

**Problem 2.**—Prove that the resistance of 1 pound of copper wire $\frac{1}{20}$ inch in diameter is sixteen times that of 1 pound of copper wire $\frac{1}{10}$ inch in diameter.

**Problem 3.**—No. 12 steel wire (American Steel Wire Gage) has a diameter of 0.1055 inch. Its specific gravity is 7.78 and its resistivity is 13.8 microhm-centimeters at 20°C.

(*a*) What will be the length of this wire, in feet, required to make a coil which will absorb 5 kilowatts at 20°C. when connected across 110-volt mains?
(*b*) How many pounds of wire will be required?

*Answer.*—(*a*) 325 feet. (*b*) 9.59 pounds.

**56. Ohms per Meter-gram.**—A method of expressing the resistivity of a conductor, which was formerly much used, and is still employed sometimes in wire specifications, is the following: The cross-sectional area of a bar or wire of uniform cross-section is equal to the volume of the bar divided by its length, and the volume of the bar is in turn equal to its mass divided by the density of the material of which it is made. Hence, using the same symbols as in the last article, and in addition calling $m$ the mass of the conductor and $\delta$ its density, or specific gravity,

$$r = \rho \frac{l}{A} = \rho\delta \frac{l^2}{m}$$

For a given material at a given temperature $\rho\delta$ is constant. Hence

$$r = k \frac{l^2}{m} \tag{2}$$

where $k$ is a constant (equal to $\rho\delta$) for a given material at a given temperature.

The resistivity of a conductor may then also be expressed indi-

rectly in terms of the factor $k$ in equation (2). When $r$ is expressed in ohms, $l$ in meters and $m$ in grams, this factor $k$ is equal to the resistance of 1 gram of the conductor made into a wire of uniform cross-section and 1 meter in length; this factor $k$ is therefore called the resistivity of the conductor in "ohms per meter-gram."

It should be noted that this method of expressing the resistivity of a conductor involves the specific gravity of the conductor, while the other methods do not    However, the determination of the resistivity of a conductor in ohms per meter-gram does not require a determination of the specific gravity, but merely the measurement of the resistance, length and mass of a given wire of the conductor, for $k = \dfrac{mr}{l^2}$. A knowledge of the specific gravity is necessary only when it is desired to convert ohms per meter-gram into microhm-centimeters, microhm-inches or into ohms per mil-foot.

The relation between resistivity in ohms per meter-gram and the other units of resistivity are as follows:

$$1 \text{ ohm per meter-gram} = \frac{10^2}{\delta} \text{ microhm-centimeters}$$

$$= \frac{39.37}{\delta} \text{ microhm-inches}$$

$$= \frac{601.5}{\delta} \text{ ohms per mil-foot}$$

where $\delta$ is the specific gravity of the conductor.

**Problem 4.**—The resistivity at 20°C. of certain samples of copper, aluminum and iron are respectively 1.72, 2.78, and 21.6 microhm-centimeters. The specific gravities are respectively 8.89, 2.70 and 7.8. What are the corresponding resistivities in microhm-inches, ohms per mil-foot, and ohms per meter-gram?

*Answer.*—

|  | Copper | Aluminum | Steel |
|---|---|---|---|
| Microhm-inches................... | 0.677 | 1.094 | 8.51 |
| Ohms per mil-foot................ | 10.34 | 16.71 | 129.9 |
| Ohms per meter-gram............. | 0.1529 | 0.0751 | 1.686 |

**Problem 5.**—To determine the resistivity of a bar of copper of uniform cross-section the following measurements were made: Current through bar, 10 amperes, voltage drop between two points $A$ and $B$ on the bar, 1.15 millivolts, distance between $A$ and $B$, 83.4 centimeters. Total length of

bar, 92.5 centimeters. Total weight of bar, 1078 grams. What is the resistivity of the bar in ohms per meter-gram?

*Answer.*—0.1606 ohm per meter-gram.

**57. Direct-current Conductance.**—The reciprocal of the direct-current resistance of a conductor is defined as the "direct-current conductance" of this conductor, *i.e.*, the direct-current conductance corresponding to a direct-current resistance $r$ is

$$g = \frac{1}{r} \qquad (3)$$

From Ohm's Law, Article 35, the potential drop produced by a current $I$ in a conductor of resistance $r$, when there is no electromotive force in this conductor, is $V = rI$. Whence, the current established by a potential difference $V$ impressed across the terminals of a conductor of conductance $g$, **when there is no electromotive force in this conductor,** is

$$I = gV \qquad (4)$$

Again, from Joule's Law, Article 35, the power dissipated as heat in a conductor of resistance $r$ when a current $I$ is established in it is $P_h = rI^2$. Whence, the power dissipated as heat in a conductor **in which there is no electromotive force,** when a potential difference $V$ is impressed across its terminals is

$$P_h = \frac{V^2}{r} = gV^2 \qquad (5)$$

The unit of conductance has the same dimensions as the reciprocal of the unit of resistance. This has led to the adoption of the term "mho" as the name for the practical unit of conductance, the word "mho" being the word "ohm" written backwards. The c.g.s. electromagnetic unit of conductance is then called the abmho. A conductance of 0.000,001 mho is called a micromho. Note that a conductance of 1 *micromho* is equivalent to a resistance of 1 *megohm*.

The "effective" conductance of a circuit to an alternating current is *not* the reciprocal of its resistance, although the relation $P_h = gV^2$ does hold (see Chapter XV).

**Problem 6.**—The insulation resistance between the conducting core and the lead sheath of a cable is 5 megohms. A non-varying potential difference of 11,000 volts is maintained between the core and the sheath.

(a) What is the conductance of this insulation?  (b) How much power is dissipated as heat in this insulation?

*Answer.*—(a) 0.2 micromho.  (b) 24.2 watts.

**58. Conductivity or Specific Conductance.**—Combining equation (3) with equation (1) shows that the conductance of a wire or bar of length $l$, of the same cross-section $S$ throughout its length, and having parallel end-surfaces, is

$$g = \frac{1}{\rho}\frac{S}{l} \tag{6}$$

That is, the conductance of a straight wire or bar is directly proportional to its cross-section and inversely proportional to its length, and directly proportional to the *reciprocal* of its resistivity.

The reciprocal of the resistivity of a substance is called the "conductivity," or "specific conductance," of the substance, and is usually represented by the symbol $\gamma$. That is, the conductivity corresponding to a resistivity $\rho$ is

$$\gamma = \frac{1}{\rho} \tag{7}$$

In terms of its conductivity the direct-current conductance of a wire or bar is then

$$g = \gamma \frac{S}{l} \tag{8}$$

**59. Per Cent. Conductivity.—Annealed Copper Standard.—** In 1862 Matthiessen found that the resistivity of the purest copper at that time obtainable was 0.141729 ohm per meter-gram at 0°C. Later determinations (1911) of the resistivity of a number of samples of commercial annealed copper wire by the Bureau of Standards gave a mean resistivity of 0.15328 ohm per meter-gram at 20°C., which, allowing for the 20° difference in temperature, agrees very closely with Matthiessen's value of 0.141729. Up until 1912 Matthiessen's value of 0.141729 ohm per meter-gram was adopted as the "standard" of resistivity, and the corresponding conductivity as the standard, or 100 per cent., conductivity. This standard of conductivity is known as Matthiessen's Standard

The standard, or 100 per cent., conductivity now in general use is the so-called "Annealed Copper Standard" adopted by the International Electrotechnical Commission in 1914. This standard is defined as follows:

1. At a temperature of 20°C., the resistance of a wire of standard annealed copper 1 meter in length and of a uniform section of 1 square millimeter is $\frac{1}{58}$ ohm = 0.017241....ohm.

2. At a temperature of 20°C., the density of standard annealed copper is 8.89 grams per cubic centimeter.

3. As a consequence, it follows from (1) and (2) that, at a temperature of 20°C. the resistance of a wire of standard annealed copper of uniform section, 1 meter in length and weighing 1 gram is $\left(\frac{1}{58}\right) \times 8.89 = 0.15328$ ohm.

By the per cent. conductivity of a conductor is meant 100 times the ratio of its conductivity to the conductivity corresponding to the Annealed Copper Standard. Therefore, the lower the per cent. conductivity the higher the resistivity of a conductor. A conductivity of $P$ per cent. is equivalent to

$$\frac{15.328}{P} \text{ ohms per meter-gram}$$

$$\frac{172.41}{P} \text{ microhms per centimeter-cube}$$

$$\frac{67.892}{P} \text{ microhms per inch-cube}$$

$$\frac{1037.3}{P} \text{ ohms per mil-foot}$$

For example, a conductivity of 60 per cent. is equivalent to a resistance of 17.29 ohms per mil-foot.

The Bureau of Standards recommends that whenever the conductivity of a sample is expressed as a percentage, the measured resistivity or conductivity be corrected to reduce it to the value it would have at 20°C. (see Article 62).

It is of interest to note that annealed copper, when very pure, may have a conductivity greater than 100 per cent. Ordinary annealed copper wire usually has a conductivity between 98 and 100 per cent., and hard-drawn copper wire a conductivity between 96 and 98 per cent. Commercial aluminum wire has a conductivity of about 62 per cent.

**60. The American Wire Gage.**—In this country, wires for electrical purposes, when less than ½ inch in diameter, are usually specified in terms of a wire gage introduced by the Brown and Sharpe Manufacturing Co. This gage is now known as the American Wire Gage, and is abbreviated A. W. G. The older abbreviation B. and S. is also used to designate this gage. Wires ½ inch in diameter, or larger, are usually specified in terms of their cross-section in circular mils.

The diameter of the wires corresponding to successive numbers on A. W. G. gage are so chosen that they differ by a constant percentage. A solid wire 460 mils in diameter is called a No. 0000 wire and a wire 5 mils in diameter is called a No. 36 wire. The next smaller size to a No. 0000 wire is No. 000, the next smaller size No. 00, the next No. 0, the next No. 1 and so on up to No. 36. The ratio of the diameters of No. 0000 and No. 36 is $\frac{460}{5} = 92$, and the ratio of the diameters of successive sizes is constant; this constant is therefore equal to the thirty-ninth root of 92. The thirty-ninth root of 92 is approximately equal to the sixth root of 2; hence the following approximate relations (since the cross-section varies as the square of the diameter and the cube root of 2 is approximately 1.26):

1. The ratio of the cross-sections of wires of successive sizes on the A. W. G. is equal to 1.26, the larger number on the gage corresponding to the smaller cross-section. This same relation holds for the weights of successive sizes for a given length.

2. The ratio of the resistances of wires of successive sizes on the A. W. G. is equal to 1.26, the larger number on the gage corresponding to the larger resistance.

3. An increase of 3 in the gage number halves the cross-section and weight, and doubles the resistance.

4. An increase of 10 in the gage number divides the cross-section and weight by 10 and multiplies the resistance by 10.

The cross-section of a No. 10 wire is approximately 10,000 circular mils, its resistance is approximately 1 ohm per 1000 feet at 20°C., and its weight is approximately 31.5 pounds per 1000 feet. From the above relations the resistance, cross-section and weight of any size of wire may be calculated approximately with but little effort. The resistance of a No. 10 aluminum wire is approximately 1.6 ohms per 1000 feet at 20°C. and its weight approximately 9.5 pounds per 1000 feet.

The above relations are for solid wire. The gage number of a stranded wire corresponds to the cross-section of the metal in it, and not to its overall diameter. The diameter of a concentric strand is approximately 15 per cent. greater than that of a solid wire of the same number; its weight and resistance are from 1 to 2 per cent. greater, depending upon the number of twists per unit length and the number of wires in the strand.

Complete wire tables, giving diameters, cross-sections, weights (usually pounds per 1000 feet or per mile), and resistances (ohms per 1000 feet or per mile), will be found in any electrical engineers' handbook. The student should familiarize himself with these tables, for he will have frequent occasion to refer to them.

**Problem 7.**—Construct a complete wire table for solid copper wire, from No. 0000 to No. 40, using the approximate relations above given, and compare with the exact wire table given in your handbook. The approximate table is to be made up of five columns, giving respectively the gage number, diameter in mils, cross-section in circular mils, pounds per 1000 feet, and ohms per 1000 feet. State the maximum error, in per cent., in the numbers in each of the columns, as compared with the numbers given in your handbook. Note particularly the temperature and per cent. conductivity for which the table in the handbook is constructed.

**61. Size of Wire Required for Direct-current Distribution Circuits.**—The simplest method of determining the size of wire required to transmit a given amount of direct-current power under specified conditions, is to calculate the resistance per 1000 feet or per mile, as explained in Chapter III, and then to find in the wire table the size of wire nearest to the calculated size. It is usual to select the size which has a resistance *less* (smaller gage number) than that calculated, unless the calculated resistance is very nearly equal to the resistance of a gage number.

A wire of a size other than that corresponding to a gage number cannot be purchased from stock, and the cost of a special size would, as a rule, be greater than the cost of the nearest standard size. It should also be noted that wires corresponding to *odd* gage numbers higher than No. 6 are practically never carried in stock, and therefore when the calculated resistance is greater than that of a No. 6 wire, the nearest *even* gage number having a resistance less than the calculated value is usually employed.

When insulated wires are used, the limiting factor in determining the size of the wire is frequently the temperature rise of the insulation, caused by the heat developed by the current in the wire. In the case of house wiring, except for very long runs of wire, this temperature rise is practically always the controlling factor.

The limiting current-carrying capacities generally adopted in this country for interior wiring are those given in the National Electric Code, issued by the National Board of Fire Under-

writers. This code, which covers various other requirements in regard to house wiring, is revised every 2 years. Copies may be obtained gratis upon application to any municipal electrical inspection bureau. Abstracts of this code, including the various tables, etc., are also usually given in electrical engineers' handbooks.

**Problem 8.**—(*a*) What is the size of wire required to transmit the load specified in Problem 4 of Article 50 under the conditions stated? (*b*) If this wire is rubber-covered and installed in-doors, will it be of sufficient size to meet the requirements of the National Electric Code? (*c*) What size wire would be necessary to meet these requirements? (*d*) With this size of wire, what will be the voltage across the terminals of the motor at maximum load, the generator terminal voltage being 120? (*e*) What will be the power lost in the transmission line, expressed as a percentage of the power delivered to the motor?

**Answer.**[1]—(*a*) No. 0. (*b*) No. (*c*) No. 0000. (*d*) 114.8 volts. (*e*) 4.53 per cent.

**Problem 9.**—Make up a table of the volts drop per 1000 feet corresponding to the current-carrying capacities given in the National Electric Code, for both rubber and "other" insulations, for all the wire sizes listed between No. 18 and 2,000,000 circular mils. This table will be found useful in determining, in any specific case, whether the current-carrying capacity or voltage drop is the limiting factor in fixing the size of wire to be used.

**62. Temperature Coefficient of Electric Resistance.**—As already noted, the resistance of a conductor depends upon the temperature of the conductor. Experiment shows that in the case of metal conductors this variation of resistance with temperature may, for all ordinary temperature changes (for example, from − 50°C. to + 200°C.), be expressed by the formula

$$r = r_1 [1 + \alpha_1 (t - t_1)] \qquad (9)$$

where $r_1$ is the resistance at the temperature $t_1$, $r$ the resistance at the temperature $t$, and $\alpha_1$ is a factor whose value depends upon (*a*) the nature of the conductor, (*b*) the value of the temperature $t_1$, and (*c*) the units in which the resistances and temperatures are expressed. This factor $\alpha_1$ is called the resistance temperature coefficient of the conductor *at the temperature $t_1$*.

The temperature of conductors is almost invariably expressed in degrees centigrade. When $t_1$ is taken as 0°C., equation (9) becomes

$$r = r_o (1 + \alpha_o t) \qquad (9a)$$

[1] See p. 1859 of PENDER's HANDBOOK FOR ELECTRICAL ENGINEERS for wire table, and p. 1908 for table of current-carrying capacities.

where $r_o$ is the resistance of the conductor at 0°C. The coefficient $\alpha_o$, which corresponds to a reference temperature of 0°C., is called the *zero-degree* temperature coefficient of resistance.

From equation (9a) the resistance $r_1$ at $t_1$°C. may then be written

$$r_1 = r_o (1 + \alpha_o t_1) \tag{9b}$$

By combining the three equations (9), (9a) and (9b), the relation between $\alpha_1$ and $\alpha_o$ is found to be

$$\alpha_1 = \frac{\alpha_o}{1 + \alpha_o t_1} \tag{10}$$

To prove this, substitute (9a) for the left-hand member of (9) and for $r_1$ in the right-hand member of (9) substitute (9b), and divide through by $r_o$. This gives

$$1 + \alpha_o t = (1 + \alpha_o t_1) [1 + \alpha_1 (t - t_1)]$$

whence

$$1 + \alpha_o t = 1 + \alpha_o t_1 + (1 + \alpha_o t_1) \alpha_1 (t - t_1)$$

and therefore

$$\alpha_o(t - t_1) = (1 + \alpha_o t_1) \alpha_1(t - t_1)$$

The cancellation of $(t - t_1)$ from both sides of the last equation gives equation (10).

The resistance temperature coefficient of a commercially pure metal is found to depend upon its degree of purity. 100 per cent. conductivity copper (Annealed Copper Standard) has a resistance-temperature coefficient equal to 0.00393 per degree centigrade at 20°C. (This is the value adopted as standard by the International Electrotechnical Commission.) The corresponding *zero-degree* temperature coefficient is 0.00427.[1] Copper of any other per cent. conductivity has a resistance temperature coefficient proportional to its conductivity.

The temperature coefficient of other commercially pure metals, except the magnetic metals (iron, nickel, and cobalt), is approximately the same as that of copper. For actual values see any electrical engineers' handbook.

The temperature coefficient of carbon is negative, and is constant over only a comparatively small range of temperature. The negative character of the temperature coefficient of carbon

[1] Prior to the year 1914, the value 0.0042 for the zero-degree temperature coefficient of copper was adopted as standard by the American Institute of Electrical Engineers.

is strikingly shown by the large decrease in the resistance of an ordinary carbon lamp when the filament is incandescent, as compared with its resistance when the filament is cold. In the case of a 110-volt, 16-candlepower lamp, the resistance changes from about 700 ohms when the lamp is cold to about 200 ohms when the lamp is burning.

Electrolytic solutions (see Article 27) also have a negative temperature coefficient, which is usually several times larger than that of metals.

The resistance of most insulating substances likewise decreases rapidly with temperature. For such substances an equation of the form of (9) is not generally applicable.

In the particular case of the rubber compounds used for cable insulations, the decrease of resistance per degree change in temperature at any temperature $t$ is found to be approximately proportional to the resistance $r$ at this temperature. This relation may be expressed mathematically by the equation

$$\frac{dr}{dt} = - kr$$

where $k$ is a constant. This equation may also be written

$$\frac{dr}{r} = - kdt$$

The integration of the two sides of this equation between the limits $r_1$ and $r$ and $t_1$ and $t$ respectively, gives

$$\log_\epsilon \frac{r}{r_1} = - k (t - t_1)$$

or

$$r = r_1 \epsilon^{-k (t - t_1)} \qquad (11)$$

where $\epsilon$ is the base of the natural logarithms.

For example, for an average grade of rubber insulation $k = 0.045$ when temperatures are expressed in degrees centigrade. Hence, if the insulation resistance of a rubber-covered cable is found to be 10 megohms at 20°C., its insulation resistance at 80°C. will be

$$r_{80} = 10\epsilon^{-0.045 \times 60} = 0.672 \text{ megohms},$$

which is a decrease of resistance of approximately 93 per cent.

In the case of paper and varnish cloth insulations the effect of

temperature changes on the insulation resistance is even greater than in the case of rubber.

The temperature coefficient of alloys depends largely upon their constituents. It is possible to make alloys for which the temperature coefficient is zero over a considerable range of temperature. Such alloys are extremely useful in the construction of standard resistance coils. The resistance of a coil made of such an alloy remains constant for any ordinary variation of temperature, and consequently does not have to be "corrected" for temperature. For values of the temperature coefficient of alloys see any electrical engineers' handbook.

Problem 10.—(*a*) What is the percentage increase in resistance of copper per degree increase in temperature, referred to the resistance at 0°? (*b*) How many degrees increase in temperature is required to increase the resistance of copper 1 per cent.?

*Answer.*—(*a*) 0.427 per cent. (*b*) 2.34° for 1 per cent. increase in resistance.

Problem 11.—The resistance of the field winding of a certain shunt motor is 17.4 ohms when the machine is at the temperature of the surrounding air, which is 21.3°C. After the machine has been running for 6 hours the average temperature of this winding is 82.1°C. The voltage impressed on the field is maintained constant at 220 volts.

(*a*) What will be the field current when the motor is first started up and what will be field current when the temperature of the winding is 82.1°C.? (*b*) What will be the percentage change in the field current, referred to its initial value?

*Answer.*—(*a*) 12.63 amperes at the start and 10.22 amperes at end of 6 hours. (*b*) The field current decreases by 19.1 per cent.

**63. Determination of the Temperature Rise of Conductors by the Change in Their Resistance.**—The change in the resistance of copper with temperature is commonly used as a means of determining the average rise of temperature of the copper windings of electric machines. When the calculations are made with a slide rule, the most convenient formula to use is that obtained by dividing (9*a*) by (9*b*) and putting

$$T_o = \frac{1}{\alpha_o} = \frac{1}{\alpha_{20}} - 20 \tag{12}$$

The ratio of the resistance at the temperature $t_1$ to the resistance at the temperature $t$ may then be written

$$\frac{r_1}{r} = \frac{T_o + t_1}{T_o + t} \tag{13}$$

The quantity $T_o$ has such a value that, were the expression $r = r_o(1 + \alpha_o t)$ applicable for all temperatures down to those approaching the absolute zero ($- 273°$C.), the resistance of the conductor at a temperature of $- T_o°$C. would be zero. Actually, however, the resistance of a conductor at a temperature equal to the negative of the reciprocal of its zero-degree temperature coefficient is not zero. For all commercially pure metals $T_o$, as defined by equation (12), is less than 273. For example, for 100 per cent. conductivity copper $\alpha_{20} = 0.00393$ therefore $T_o = 234.5$. The quantity $- T_o$, where $T_o$ is defined by equation (12), is called the "inferred" absolute zero of the conductor.

For copper of 100 per cent. conductivity, which is very closely that of ordinary annealed copper wires, equation (13) becomes

$$r_1 = \left(\frac{234.5 + t_1}{234.5 + t}\right) r \qquad (13a)$$

In slide-rule calculations, the figure 234, instead of 234.5, is sufficiently accurate.

The operation expressed by this equation may be readily performed by one setting of a slide rule, when a special temperature scale is marked off on the lower scale of the slide, with 234 as 0°, 244 as 10°, 254 as 20°, etc. If the resistance at 25° is 3 ohms, say, the resistance at any temperature, say 60°, is then found by setting 25 on this new scale opposite 3 on the lower scale of the rule, and reading off on the lower scale the number corresponding to 60 on the new scale, *i.e.*, 3.41 ohms.

When the two resistances $r_1$ and $r_2$ and the room temperature $t$ are known, the value of $t_1$ may be readily calculated by the reverse process. For example, if the resistance of a coil at 20°C. is found to be 5 ohms and its resistance when heated, 6 ohms, then by setting 20 on the temperature scale of the slide rule opposite 5 on the lower scale, the value of $t_1$ is given on the temperature scale opposite 6 on the lower scale, namely, 71°C. The average rise of temperature of the coil is then $71 - 20 = 51°$.

**64. Resistance of Non-linear Conductors.**—Equation (1) for the relation between the resistance and the dimensions of a conductor applies only under the specific conditions stated in Article 54. When the lines of flow of the electric current through a conductor are not straight, parallel and uniformly distributed, equation (1) is not directly applicable. However, by using the

relation expressed by equation (1) as a starting point, formulas can be derived which are applicable irrespective of the shape of the conductor and of the points at which the current enters and leaves it, as will now be shown.

A surface every point of which is at the same electric potential, *i.e.*, between any two points of which no difference of electric potential exists, is called an "electric equipotential surface." Between any two equipotential surfaces between which a finite difference of potential exists, there must exist an infinite number of equipotential surfaces, each separated from the next by an infinitesimal distance and differing in potential therefrom by an infinitesimal amount. Since the electric potential at any point
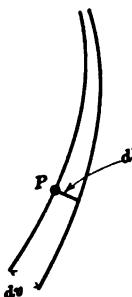


FIG. 34.

can have but a single value, no two of these equipotential surfaces can intersect each other.

Imagine two equipotential surfaces which are infinitely close together, and let $dl$ be the perpendicular distance between these two surfaces at any point $P$. Let $dv$ be the difference of potential between the two surfaces. Then the quotient

$$\frac{dv}{dl}$$

is called the "electric potential gradient" at the point from which $dl$ is drawn.

The potential drop along a given distance divided by this distance is equal to the potential drop per unit length; hence, the potential gradient at any point may be defined in words as the *drop of potential per unit length* at this point, it being understood that the length is infinitely small and is perpendicular to the equipotential surface through this point.

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Experiment justifies the assumption that the flow of electricity past any point in a body is always perpendicular to the equipotential surface through that point. A line drawn in a body tangent at each point to the direction of the flow of electricity past that point, is called a "line of flow" of electricity, or briefly, a "stream line" of the electric current. The stream lines of an electric current and the electric equipotential surfaces are therefore always mutually perpendicular. In Fig. 35 are shown, in two dimensions only, the electric equipotential surfaces (dotted) and the stream lines in a carbon block, when the current is led into and out of the block through two small terminals located
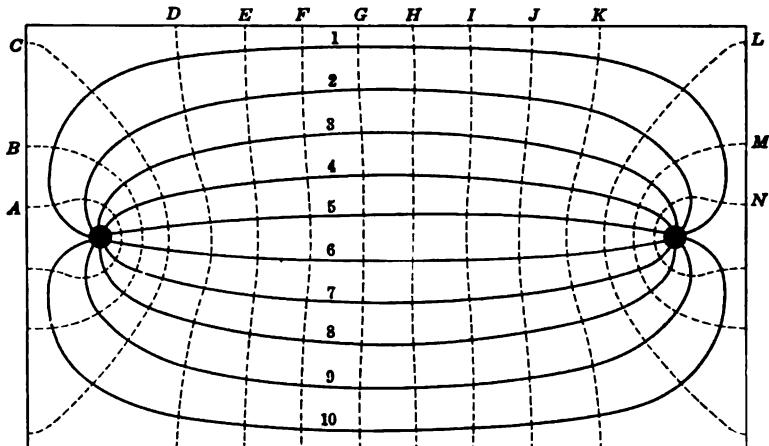


FIG. 35.—Stream lines and equipotential surfaces.

at the black dots in the figure. The equipotential surfaces were actually determined experimentally by means of a potentiometer.

The "density" of the current at any point in a conductor is defined as the current *per unit area* which flows through an elementary area at this point taken normal to the direction of the stream line through this point. Such an area is part of an equipotential surface. Let $dS$ be such an area at any point $P$ in a conductor, and let $di$ be the current through this area, then the current density at the point $P$ is

$$\sigma = \frac{di}{dS} \tag{14}$$

The distribution of the current in a conductor may be conveniently represented graphically by drawing in this conductor

such a number of stream lines that their number per unit area normal to their direction is equal, for each such area in the conductor, to the current density at this area. The stream lines shown in Fig. 35 are so drawn.

Any region in which a potential gradient exists may be thought of as divided into cells (like a honeycomb) whose ends are formed by equipotential surfaces and whose side walls are formed by surfaces perpendicular to the equipotential surfaces, and therefore parallel to the direction of the electric current, if any, in these cells. These cells may be thought of as being as small as one wishes, and in the limit each cell may be considered a right cylinder with its axis perpendicular to the equipotential surface which forms either end. Each of these infinitesimal cells then satisfies the conditions necessary for the application of equation (1), Article 55.

Consider any such infinitesimal cell at any point $P$ in a conductor, and let $dl$ be its length and $dS$ its cross-section, and let $\rho$ be the resistivity of the conductor at this point. Then the resistance of this cell is

$$r = \rho \frac{dl}{dS}$$

Let $\sigma$ be the current density at the point $P$. Then the current through this cell is

$$di = \sigma dS$$

Hence the drop of potential through this cell, due to its resistance, is

$$dv_r = rdi = \rho\sigma dl$$

The resistance drop *per unit length* at the point $P$ is then

$$\frac{dv_r}{dl} = \rho\sigma$$

and its direction coincides with the direction of the current, or flow of electricity, at this point.

Note that drop of potential has the dimensions of work per unit quantity of electricity, or work per unit charge, and work has the dimensions of force times distance. Consequently, the resistance drop per unit length, namely, $\rho\sigma = \frac{dv_r}{dl}$, has the same dimensions as force per unit charge. The resistance drop per

unit length at any point is therefore equal to the *force per unit charge* which acts on the electricity at this point, tending to move it through the medium in which this point is located. This force per unit charge is called the "intensity of the electric field," or briefly, the "electric intensity," at this point. The electric intensity (*i.e.*, the intensity of the electric field) at any point may be represented by the symbol $F$. The electric intensity at any point is then, by definition,

$$F = \rho\sigma \tag{15}$$

where $\sigma$ is the current density at this point and $\rho$ is the resistivity of the medium at this point.[1]

The direction of the electric intensity at any point is defined as the direction of the flow of electricity at this point, *i. e.*, the electric intensity at a given point has the direction of the stream line of the current through that point.

From the relations just established it is evident that when the current density at each point in a conductor is known, or can be calculated, the resistance of this conductor between any two equipotential surfaces $A$ and $B$ can be readily expressed. For, if $\sigma$ is the current density and $\rho$ the resistivity at any point, the resistance drop in a length $dl$ of the stream line of the current at this point is

$$dv_r = F dl = \rho\sigma dl$$

Therefore the total resistance drop from the equipotential surface $A$ to the equipotential surface $B$ is

$$v_r = \int_A^B F dl = \int_A^B \rho\sigma dl \tag{16}$$

Hence, calling $i$ the total current in the section of the conductor between these two equipotential surfaces, the resistance of this section of the conductor is

$$r = \frac{v_r}{i} = \frac{\int_A^B \rho\sigma dl}{i} \tag{17}$$

[1] In this definition $\sigma$ is the density of the *conduction* current at the given point; the *displacement* current is not included (see Article 137).

**Problem 12.**—The carbon block shown in Fig. 35 is 4¾ inches long, 2.8 inches wide, and ¾ inch thick. The total current through the block was 5 amperes, and each stream line therefore represents 0.5 ampere. The total drop of potential between the two terminals was 30 millivolts, and the equipotential surfaces are so drawn that they are 2 millivolts apart. Each terminal (each black dot) may be considered an equipotential surface.

(a) What is the current density at the center of the block?  (b) What is the current density at the equipotential surface A where it intersects the two central stream lines?  (c) What is the potential gradient at the center of the block?  (d) What is the average potential gradient along the center line of the block between one terminal and the equipotential surface next to it?  (e) What is the average potential gradient along the center line of the block between the two equipotential surfaces A and B?  (f) What is the resistivity of the block?  (g) What is the total resistance of the block between the two terminals?  (h) What would be the resistance of the block between two parallel troughs of mercury, perpendicular to the axis of the block and extending all the way across it, and at the same distance apart as the round terminals shown?  (NOTE.—The calculated resistances in (g) and (h) are the resistances of the carbon itself, and not the *total* resistance between terminals, which latter includes the "contact" resistance between the mercury and the carbon.)  (i) What would be the shape of the stream lines and equipotential surfaces in (h)?

*Answer.*—(a) 2.38 amperes per square inch.  (b) 8.3 amperes per square inch.  (c) 6.2 millivolts per inch.  (d) 25 millivolts per inch.  (e) 14 millivolts per inch.  (f) 0.0026 ohm-inch.  (g) 0.0060 ohm.  (h) 0.0040 ohm.  (i) The stream lines would be straight lines parallel to the axis of the block, and the equipotential surfaces would be planes perpendicular to these lines.

## 65. Insulation Resistance of a Single-conductor Cable.—By making use of the relations developed in the last article, the resistance of non-linear conductors of certain symmetrical shapes may be readily calculated. For example, the insulation resistance of a single-conductor cable, shown in cross-section in Fig. 36, may be expressed in terms of the diameter $d$ of the conducting core, the external diameter $D$ of the insulation around it, the resistivity $\rho$ of this insulation, and the length $l$ of the cable. The insulation of such a cable is frequently covered with a lead sheath as shown. When such a sheath is not provided, the insulation resistance is tested by immersing the cable in water, which latter may then be considered as forming a conducting sheath around it.

Let $V$ be difference of potential established between the core and the sheath which surrounds the insulation, and let $I_l$ be the current, in amperes, which flows through the insulation from the core to the sheath. This current is usually referred to as the

·"leakage" current, and is not to be confused with the main current which flows through the core to the load to which the cable may be connected.

Due to the high conductivity of the core, the drop of potential along it will be negligibly small in comparison with the drop of potential through the insulation, and therefore the surface of the core may be considered an equipotential surface. In the sheath there will be no other current than the leakage current, and due to the relatively high conductivity of the sheath, the external surface of the insulation may likewise be considered an equipotential surface.

The stream lines of the leakage current through the insulation must therefore be perpendicular to both the surface of the core and to the external surface of the insulation, and when the insu-
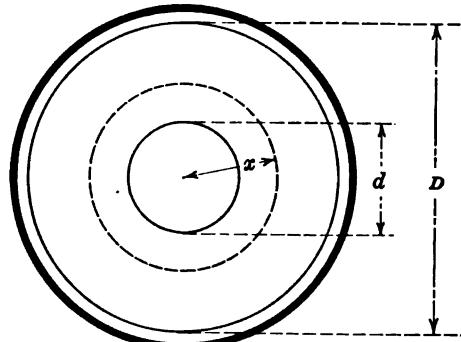


Fɪɢ. 36.

lation is of uniform thickness, must, from symmetry, be equally spaced radial lines. Hence the current density at any point in the insulation at a distance $x$ from the center of the wire is equal to the total leakage current $I_l$, divided by the area of the cylindrical surface which passes through this point and which is concentric with the wire, viz.,

$$\sigma = \frac{I_l}{2\pi x l}$$

Whence, from equation (16),

$$V = \frac{\rho I_l}{2\pi l} \int_{x=\frac{d}{2}}^{x=\frac{D}{2}} \frac{dx}{x} = \frac{\rho I_l}{2\pi l} \log_{\epsilon} \left(\frac{D}{d}\right)$$

Therefore, the insulation resistance of the given length of cable is

$$R = \frac{V}{I_l} = \frac{\rho}{2\pi l} \log_e \left(\frac{D}{d}\right) \tag{18}$$

Note that the insulation resistance varies inversely as the length. This is evidently true when it is remembered that each elementary length of the insulation of the cable (measured along its axis) is in parallel with all the other elementary lengths.

In equation (18) the length $l$ must be expressed in centimeters when $\rho$ is expressed in ohm-centimeters, in order that $R$ be in ohms. The two diameters $D$ and $d$ may be expressed in any unit of length, provided both are expressed in the *same* unit, for it is their ratio only which enters into the formula.

Cable manufacturers usually express the resistivity of cable insulation in terms of the constant $K$ defined by the equation

$$R_{60} = K \log_{10} \left(\frac{D}{d}\right) \tag{18a}$$

where $R_{60}$ is the insulation resistance, in megohms at 60°F., of a cable 1 mile long, and $D$ and $d$ are respectively the external diameter of the insulation and the diameter of the core.

Note that $R_{60}$ megohms is equal to $10^6 R_{60}$ ohms, 1 mile $= 2.54 \times 12 \times 5280 = 1.609 \times 10^5$ cm., and $\log_e = 2.303 \log_{10}$. Hence, from equation (18), if $\rho$ is the resistivity in ohm-centimeters, then

$$R_{60} = \frac{\rho \times 2.303}{2\pi \times 1.609 \times 10^5 \times 10^6} \log_{10} \left(\frac{D}{d}\right)$$

$$= 2.28 \times 10^{-12} \rho \log_{10} \left(\frac{D}{d}\right)$$

Whence

$$K = 2.28 \times 10^{-12} \rho \tag{19}$$

For ordinary rubber insulation $K$ usually has a value between 3000 and 8000, for impregnated paper $K$ usually has a value between 1000 and 2000, and for varnished cambric $K$ usually has a value between 700 and 1200. These values of $K$ correspond to the ohmic, or direct-current, resistance of the insulation. The "effective" alternating-current resistance is usually much less (see Chapter XV).

**Problem 13.**—Two single-conductor lead-covered rubber-insulated cables placed side by side (with their sheaths in contact) are used to supply direct-current electric energy from a power-house to a substation 10 miles away. The conductor in each cable is stranded and has a cross-section of 250,000 circular mils (mean diameter 575 mils) and the thickness of the insulation is $\frac{9}{64}$ inch. The average line voltage between the conductors in the two cables is 3000. The value of $K$ for the insulation is 5000 (at 60°F.) and the average temperature of the insulation is 100°F. The rate of decrease of the insulation resistance with temperature is 0.025 ohm per degree Fahrenheit per ohm resistance (see equation (11), Article 62).

(a) What is the insulation resistance of each cable? (b) What will be the total leakage current through the cable insulation? (c) What will be the total power dissipated as heat in the cable insulation? (d) What is the resistivity of the cable insulation at 60°F. and at 100°F.?

*Answer.*—(a) 31.9 megohms at 100°F. (b) 0.047 milliampere. (c) 0.141 watt. (d) 2190 × 10$^{12}$ ohm-centimeters at 60°F., and 805 × 10$^{12}$
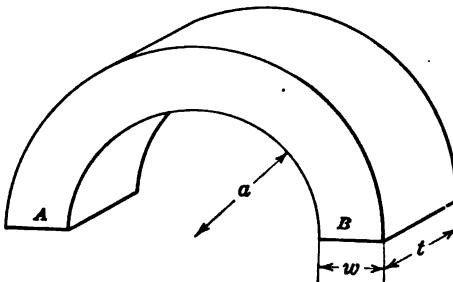


Fig. 37.

ohm-centimeters at 100°F. (Note.—The values given in this problem are typical of practical conditions, and bring out strikingly the negligible magnitudes of the leakage current and loss occasioned thereby, as compared with the load current and power delivered over a line used for power transmission. In telephony and telegraphy, however, the leakage current is by no means negligible, for the line current in a telephone or telegraph line is itself usually but a fraction of a milliampere, or at most but a few milliamperes.)

**Problem 14.**—Prove that the resistance of a uniformly tapered wire of circular cross-section (*i.e.*, a wire whose diameter changes by a constant amount per unit distance measured along its axis) is

$$r = \rho \frac{4l}{\pi d_1 d_2} \text{ ohms,}$$

where $\rho$ is the resistivity of the wire in ohm-centimeters, $l$ its length in centimeters and $d_1$ and $d_2$ the diameters, in centimeters, of its cross-section at its two ends. State what assumption is made in regard to the stream lines and equipotential surfaces, and justify this assumption.

**Problem 15.**—Fig. 37 represents a conductor of rectangular cross-section bent to form half a circular ring. The internal radius of the ring is $a$, the

width of the ring is $w$, and the thickness of the ring $t$, all in centimeters. Let $\rho$ be the resistivity of the conductor in ohm-centimeters.

(a) Prove that the resistance of this ring, when its two end-surfaces $A$ and $B$ are equipotential surfaces, is

$$r = \rho \frac{\pi}{t \log_\epsilon \left(1 + \dfrac{w}{a}\right)}$$

(b) Prove that, were the formula $r = \rho \dfrac{l}{S}$ assumed to be applicable to such a ring, when for $l$ is taken the distance measured from one end-surface to the other along the mean circumference, the error in the calculated resistance of the ring would be

$$100 \left[ \frac{2a + w}{2w} \log_\epsilon \left(1 + \frac{w}{a}\right) - 1 \right] \text{ per cent. high.}$$

(c) What would be the per cent. error in the resistance if calculated by the formula $r = \rho \dfrac{l}{S}$ for a semicircular ring when (1) the internal radius of the ring is equal to one-half the width, (2) equal to the width, (3) equal to five times the width? (d) What will be the relative current densities at a point on the inside and outside surfaces of the ring under the three conditions specified in (c)?

*Answer.*—(c) 10 per cent., 4 per cent. and 0.3 per cent. (d) The current density at the inside surface will be respectively 3, 2 and 1.2 times the current density at the outside surface.

# V

## ELECTROLYTIC CONDUCTION

**66. General.**—The electric resistance of a given volume of an electrolyte depends upon the size and shape of this volume and upon the location and shape of the electrodes at which the current enters and leaves it, in the same manner as the resistance of a given volume of a metallic conductor depends upon the size and shape of this volume and upon the location and shape of the terminals at which current enters and leaves it (see Article 64).

The conductivity of an electrolytic solution in general increases with its concentration up to a certain point, and then decreases with a further increase in concentration. For example, at 18°C. a 5 per cent. aqueous solution of sulphuric acid has a resistivity of 4.80 ohm-centimeters, a 40 per cent. solution a resistivity of 1.47 ohm-centimeters, and a 70 per cent. solution a resistivity of 4.64 ohm-centimeters.

The resistance of an electrolyte, whether a solution or fused salt, decreases with increase in temperature; *i.e.*, the temperature coefficient is negative. The resistance-temperature coefficient of aqueous solutions diminishes rapidly with increase of temperature. For values of both the conductivity and temperature coefficient see any electrical engineers' handbook.

As explained in Article 27, the flow of electricity through an electrolyte is always accompanied by a chemical action at the electrodes. This chemical action may be accompanied by either a gain or loss of chemical energy and a corresponding loss or gain of electric energy. This transformation of energy may be looked upon as due to the electromotive forces at the junctions between the electrodes and the electrolyte, and is of an entirely different nature from the dissipation of heat within the electrolyte due to its resistance.

**67. Faraday's Laws of Electrolysis.**—The chemical changes which take place at the electrodes in an electrolyte were investigated by Faraday in the early part of the last century. The

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results of Faraday's investigations are summarized in the following two laws, known as Faraday's Laws of Electrolysis:

(*a*) The mass of the chemical element or radical which is deposited or liberated at either electrode, when a current is passed through an electrolyte, is directly proportional to the total *quantity* of electricity which flows through the electrolyte.

(*b*) When the same current is established through two or more different electrolytes, the masses of the several elements or radicals deposited in a given time are directly proportional to the chemical equivalent of these elements or radicals.

By the chemical equivalent of an element is meant its atomic weight divided by its valence. By the chemical equivalent of a radical is meant its "formula" weight divided by its valence. By the formula weight of a radical is meant the sum of the atomic weights of the elements of which it is formed, for example, the formula weight of $NO_3$ is $14.01 + 3 \times 16 = 62.01$. By the valence of an element or radical is meant the number of hydrogen atoms, or one-half the number of oxygen atoms, with which 1 atom of the element, or the atomic group in the case of a radical, forms a stable combination. The valence of hydrogen is therefore 1, of oxygen 2, of copper 2 in cupric compounds and 1 in cuprous compounds, etc.

For direct currents, Faraday's two laws may be expressed mathematically as follows: Let

$I$ = current through the electrolyte, in amperes.

$t$ = time, in seconds, during which this current flows.

$M$ = total mass, in grams, of the ions liberated or deposited at an electrode.

$w$ = atomic weight, or formula weight, of these ions

$n$ = valence of these ions.

Then, since

0.001118 = weight in grams of the silver deposited in 1 second by 1 ampere, and

107.93 = atomic weight of silver,

the total mass of the given ions deposited in the time $t$ is

$$M = kIt \tag{1}$$

where

$$k = \frac{0.001118\, w}{107.93\, n} = \frac{w}{96{,}540\, n} \tag{2}$$

The constant $k$ defined by equation (2) is called the "electrochemical equivalent" of the given ions, and the constant 96,540 is called the "Faraday." Tables of electrochemical equivalents will be found in any book of physical tables or in any electrical engineers' handbook.

A mass of ions in grams equal to the formula weight of these ions is called a gram-ion. An easily remembered definition of the Faraday (*i.e.*, the constant 96,540 in equation (2)) is that the Faraday is the number of coulombs required to deposit 1 gram-ion of univalent ions.

Faraday's two laws are found to be strictly true only when proper corrections are made for all secondary reactions which may take place at the electrodes and within the electrolyte. From the practical standpoint these corrections are ordinarily relatively small. However, when the chemical action of a current is used as a basis for the absolute measurement of current (as in the definition of the international ampere, see Article 32), corrections for all secondary effects must be carefully made, or else some standard specification for the construction and use of the voltameter must be adopted.

Problem 1.—A non-varying current is passed through a solution of cupric sulphate for 1 hour and 25 minutes. Both the anode and cathode are copper plates. During this interval 15 grams of copper are deposited from the electrolyte on one of the electrodes and the other electrode wastes away. What is the value of the current in amperes, and on which electrode is the copper deposited? (Note.—The atomic weight of copper is 63.57, and its valence in a cupric salt is 2.)

*Answer.*—8.93 amperes. Copper is deposited on the cathode.

**68. Dissociation Theory of Electrolytes.**—A relatively simple explanation of electrolytic conduction, as well as of certain other physical and chemical characteristics of electrolytes, is possible on the assumption that each molecule of an electrolyte, when dissolved, breaks up into ions[1] which exist in a more or less unstable condition of equilibrium, and that the anions contain an excess of · negative electricity, and the cations an equal excess of positive electricity, *i.e.*, that the anions and cations are equally and oppositely charged. On this assumption, the force which causes the electric current to flow through the electrolyte, when the latter is connected in series with a battery or other source of

[1] The word "ion" is here used to designate the single particles into which the molecule is assumed to break up.

electromotive force, causes the cations (metal or hydrogen) to move in the direction of the current and the anions to move in the opposite direction.

Moreover, on the basis of the further assumptions, which are in accord with all known experimental facts, that (1) each ion of a given kind has a constant mass, (2) that the charge on a single ion of any given kind is constant, and (3) that the charges on single ions of different kinds are proportional to their valence, the quantitative relations known as Faraday's Laws may be readily deduced.

The assumption that the charge on an ion is proportional to its valence arises from the fundamental assumption that a neutral molecule contains equal and opposite charges. For example, if each hydrogen atom in a molecule of sulphuric acid ($H_2SO_4$) carries $q$ units of positive charge, the $SO_4$ radical must carry $2q$ units of negative charge, in order that the molecule as a whole may manifest none of the properties of either kind of charge.

The deduction of Faraday's Laws from the above assumptions is then as follows: Let

$q$ = charge on a single hydrogen atom in any electrolyte.
$m$ = mass of a single ion in the electrolyte under consideration.
$n$ = valency of this ion.
$N$ = number of single ions which are liberated at the electrode (cathode or anode, depending upon the nature of the ion).

Then the total mass liberated at this electrode is $M = Nm$, and the total quantity of electricity transferred to this electrode by the cations is $nNq$. Since the electrode does not become charged, this electricity must flow through the electrode into the external circuit, or an equal quantity of electricity of the opposite sign must flow up to this electrode through the external circuit. In either case, the quantity $nNq$ must be equal to the product of the current $I$ in the circuit by the time $t$ during which this current flows. Hence $nNq = It$, or $N = \dfrac{It}{nq}$, and therefore

$$M = \frac{m}{nq} It \qquad (3)$$

Hence, since $q$ is a fixed constant and $m$ and $n$ are also constants for any particular ion, the total mass of the ions deposited or

liberated at the electrode is proportional to the total quantity of electricity transferred through the circuit.

The factor of proportionality,

$$k = \frac{m}{nq} \tag{4}$$

is the electrochemical equivalent of the ion. A comparison of this expression with equation (2) shows that the ratio of the charge carried by a given ion to the mass of this ion is equal to 96,540 times the ratio of its atomic (or formula) weight to its valency. For example, a hydrogen atom carries a charge, in coulombs, equal to 96,540 times its mass in grams.

## SOURCES OF ELECTROMOTIVE FORCE

**69. General.**—As explained in Article 24, any device which produces, or tends to produce, a flow of electricity through a conductor which is connected between its two terminals, is said to be a source of electromotive force.

As explained in Article 38, the *quantitative measure* of the electromotive force of a device which gives out electric energy is defined as the electric power developed within the device per unit of current through it. Also, when there is no current through a device, its electromotive force is defined quantitatively as the limiting value of the quotient

$$\frac{\text{electric power developed within the device}}{\text{electric current through the device}}$$

when the intensity of the current through it approaches the value zero.

As explained in Article 40, a difference of potential in general exists between the terminals of a source of electromotive force and, when there is no current through the device, its electromotive force and terminal potential difference are numerically equal.

Experiment shows that the two primary sources of electromotive force are:

1. Two dissimilar substances in contact, the dissimilarity being either in their chemical nature, temperature or concentration (in case either is a solution).

2. A conductor which is in the vicinity of a conductor in which a varying current is flowing, or between which and a magnet or any conductor carrying a current (constant or variable) there exists a relative motion.

Electromotive forces of the first kind are the simplest to describe and correlate, but from the standpoint of engineering, the electromotive forces of the second kind are by far the more important. The relation between these latter electromotive

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forces and the effects which produce them can best be expressed in terms of a quantity called "magnetic flux," which will be considered in detail in the next chapter.

**70. Contact Electromotive Forces at Junctions Between Metals.**—The electromotive force at the junction between two dissimilar substances is called a "contact" electromotive force. Experiment shows that the contact electromotive force at such a junction has a definite magnitude and direction, which depend only upon (1) the chemical nature of the two substances, (2) the temperature of the junction, and, in case either is a solution, upon (3) the concentration of the solution. A contact electromotive force does not depend upon the area of the contact surface or upon the size of the bodies in contact.

In the case of two metals in contact, this contact electromotive force manifests itself, when a current is passed through the junction, by the heating or cooling of the junction. An increase in the temperature of the junction indicates a transformation of electric energy into heat energy; a decrease in the temperature of the junction indicates a transformation of heat energy into electric energy. Either of these two effects may be produced, depending upon the direction of the current through the junction, *i.e.*, the electromotive force at the junction has a definite direction with respect to the two metals in contact. This heating or cooling of a junction in consequence of its contact electromotive force is usually inappreciable in comparison with the heating of the conductors due to their ohmic resistance, but when special precautions are taken the effect described may be readily detected.

The value of the contact electromotive force between metallic conductors is quite small, being only a small fraction of a volt (*e.g.*, between copper and zinc at 25°C. it is 0.00045 volt), and in practical work it is therefore usually negligible.

It is found by experiment that whenever any number of metallic conductors are connected in series, so that they form a closed loop, the algebraic sum of the contact electromotive forces at the several **metal-metal** junctions in this loop is zero, **provided all the conductors are kept at the same uniform temperature.** As a consequence of this law, it follows that in a closed chain of two or more dissimilar metals, all at the same temperature, the resultant electromotive force due to the contact electromotive forces at the several metal-metal junctions is zero, and

therefore no current will flow in such a circuit unless there is some other source of electromotive force present.    This fact is sometimes called the Law of Successive Contacts.

For example, when two copper wires are soldered together, the net effect of the solder will be nil, whether the two copper wires are in actual contact or are joined only through the solder, provided the flux used with the solder does not contain an acid or other electrolyte.    If an acid flux is used, sufficient acid may be left in the junction to form a minute voltaic cell (see Article 72), the electromotive force of which may be appreciable, particularly when the junction is in the circuit of an instrument employed for the measurement of potential differences of the order of a volt or less.

**71. Thermal Electromotive Forces.**—The Law of Successive Contacts, which applies only to metal-metal junctions, holds only when all the junctions are at the same temperature.    When any one junction is kept at a higher temperature than another, the resultant of the several contact electromotive forces will not in general be zero, but will have a value depending upon the difference in the temperatures of the junctions.    This fact is made use of in the construction of thermocouples, or electric pyrometers, which have an important practical application in the measurement of high temperatures.

The ordinary form of electric pyrometer consists essentially of a platinum and a platinum-rhodium wire fused together at one end and connected in series with a millivoltmeter (*i.e.*, a voltmeter designed to measure differences of potential of the order of a thousandth of a volt).    The junction between the two wires, suitably protected by a porcelain or quartz tube, is placed in the furnace the temperature of which it is desired to measure.    The difference of potential indicated by the voltmeter is then practically proportional to the difference between the temperature of the hot junction in the furnace and the temperature of the "cold junction" (*i.e.*, the junction between the two ends of the thermocouple wires and the leads to the voltmeter).

An electromotive force of only a small fraction of a volt can be obtained from a single pair of junctions, even when the two junctions are at temperatures differing by 1000°C. or more.    However, by connecting in series alternate lengths of the two dissimi-

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lar metals, so arranged that alternate junctions may be heated and the intermediate junctions kept cool, a resultant electromotive force equal to the sum of the electromotive forces of the several pairs of junctions may be obtained; this is the principle of the "thermopile."

It is also of interest to note in this connection that an electromotive force, usually of negligible magnitude, however, is also produced even in a conductor of uniform chemical nature throughout, unless this conductor is kept at a uniform temperature throughout. This phenomenon is known as the Thomson effect.

**72. Electromotive Force of Voltaic Cells.**—In the case of a metal in contact with an electrolyte (*e.g.*, dipping into it), experiment shows that the contact electromotive force is of a relatively large magnitude, being of the order of 1 volt or more, and further, that when there are two dissimilar metals dipping into the same solution, the resultant of the two contact electromotive forces is not zero. The electromotive force of a simple voltaic cell is the resultant of the contact electromotive forces at the two metal-liquid junctions within it.

For example, in the case of copper-sulphuric-acid-zinc cell, the electromotive force of contact between the zinc and the sulphuric acid solution is in the direction from the zinc to the acid and is about 1 volt greater than the contact electromotive force between the copper and the acid, which latter electromotive force is in the direction from the copper to the acid. Hence, the resultant electromotive force of the cell is about 1 volt, and is in the direction through the battery from the zinc to the copper pole. The copper pole is therefore at a potential about 1 volt higher than that of the zinc pole.

The electromotive force of a voltaic cell depends not only upon the chemical nature of the two metals, but also upon the chemical nature and concentration of the electrolyte in contact with these metals, and upon the temperatures of these junctions. In general, when an electric current is established through a cell, the concentration of the electrolyte changes, and some of the products of the chemical reactions collect at the electrodes, and consequently the electromotive force of the cell changes. This phenomenon is usually described by saying that the cell becomes "polarized."

The polarization of a voltaic cell is usually due primarily to the formation of hydrogen gas, or to the liberation of metal, at the positive pole. By placing in the electrolyte around the positive pole an oxidizing agent, such as manganese peroxide (see Fig. 10), the hydrogen or metal liberated at the positive pole may be reduced to an acid or salt readily soluble in the electrolyte, and its accumulation around the pole will be less rapid. Such an oxidizing agent or other substances used for the purpose of preventing or decreasing the polarization of a cell is called a "depolarizer."

The polarization of a dry battery is particularly noticeable, in spite of the depolarizer which it contains, since the electrolyte in such a cell, instead of filling the entire space between the poles, simply impregnates the practically solid mass between them. However, when such a battery is left open-circuited for a time after it is used, the products of the chemical reactions which take place when the cell is discharging, gradually diffuse through the mass between the two poles, and the electromotive force of the cell gradually returns to practically its original value, provided none of the active materials has been completely destroyed.

Certain special forms of voltaic cells are substantially reversible in their action; *i.e.*, when an electric current is sent through such a cell in one direction, electric energy is stored in the cell, in the form of chemical energy, and when an electric current is allowed to flow through the cell in the opposite direction, electric energy is given out. As noted in Article 37, such a cell is called a "storage" cell. Other names used to designate such a cell are "secondary" cell and electric "accumulator."

In order that a voltaic cell may act as a storage cell, it is necessary that when a current is sent through it in the opposite direction to that in which it would of itself establish a current, the chemical action which takes place at its electrodes be substantially the reverse of the chemical action which takes place when the cell itself produces the current. The commonest form of storage cell is the lead-sulphuric-acid cell described in Article 37. Another form of storage cell is the Edison cell, which has electrodes of iron and nickel, and in which the electrolyte is a solution of caustic potash.

Due to the phenomenon of polarization, the electromotive force of a storage cell when discharging is always less than its

"open-circuit" electromotive force; and when charging, its electromotive force is always greater than its "open-circuit" electromotive force. Due to the internal resistance of the cell, the variation in its terminal voltage is of course greater than the variation in the internal electromotive force. The open-circuit electromotive force of a lead storage cell is about 2.2 volts, and the open-circuit electromotive force of an Edison cell is about 1.5 volts. Although the electromotive force of an Edison cell is less than that of a lead cell, the watt-hour capacity per pound of total weight is greater for the Edison cell than for the lead cell.

For a description of the various kinds of voltaic cells in common use, both primary cells and storage (or secondary) cells, the student is referred to any comprehensive electrical engineers' handbook.

**73. Standard Cells.**—Many of the ordinary forms of voltaic cells undergo chemical changes in their various parts even when left open-circuited. This is due chiefly to impurities in the chemicals of which they are made. It is possible, however, to construct a cell which will remain practically unaltered for several years, provided no current is taken from it. Hence, such cells make a very convenient standard of electromotive force or potential difference. Two such standard cells are in general use, namely, the Clark cell and the Weston cell. The Weston cell has a longer life and polarizes less rapidly than the Clark cell.

In the Clark cell, the two electrodes are mercury and zinc amalgam, the electrolyte is zinc sulphate, and a paste of mercurous sulphate and zinc sulphate is used as a depolarizer. Detailed directions for the preparation of the cell are given in the *Bulletin* of the Bureau of Standards, 1907, vol. 4, p. 1. The electromotive force of the cell constructed in accordance with these specifications is 1.4328 international volts at 15°C. At any other temperature of $t$°C., the electromotive force of this cell is

$$1.4328 \, [1 - 0.00077 \, (t - 15)] \text{ international volts}$$

In the Weston cell, the electrodes are mercury and cadmium amalgam, the electrolyte is cadmium sulphate, and a paste of mercurous sulphate and cadmium sulphate is used as a depolarizer. Detailed directions for the preparation of this cell are given in the *Bulletin* of the Bureau of Standards, 1907, vol. 4, p. 1. The electromotive force of this cell when constructed in accord-

ance with these specifications is 1.01830 international volts at 20°C.   At any other temperature of $t$°C., its electromotive force is

$$1.01830 - 10^{-6}[40.6\,(t-20) + 0.95\,(t-20)^2 + 0.01\,(t-20)^3]$$

See the *Bulletin* of the Bureau of Standards, 1909, vol. 5, p. 309.

In the practical use of either of these cells as a standard of comparison of potential differences, an arrangement of circuits should be employed which will obviate the necessity of taking any current from the cell; for example, a potentiometer (see Article 45).

## MAGNETIC FLUX AND THE MAGNETIC CIRCUIT

**74. Electromagnetic Induction.**—The production of electric energy from chemical energy, by means of a voltaic cell, was discovered in the eighteenth century. The production of large amounts of electric energy by such a device, however, is not commercially feasible, due to the high cost of the materials required per unit of energy thus produced. Commercial generators of electric energy are based upon an entirely different class of phenomena, discovered independently by Michael Faraday, an Englishman, and Joseph Henry, an American, between 1830 and 1832.

These phenomena are briefly as follows:

(a) When a **magnet is in motion with respect to an electric conductor** in its vicinity, an electromotive force is in general produced in this conductor, irrespective of whether or not there is a current in it.

(b) Conversely, when an **electric conductor is in motion with respect to a magnet** in its vicinity, an electromotive force is in general produced in this conductor, irrespective of whether or not there exists a current in it.

(c) When a **conductor in which an electric current exists is in motion with respect to another conductor** in its vicinity, an electromotive force is in general produced in the second conductor, irrespective of whether or not a current exists in this second conductor.

(d) When the **intensity of the electric current in a conductor is changing with time,** an electromotive force is produced in this conductor and, in general, in every other conductor in its vicinity, irrespective of whether or not there exists a current in the latter conductor.

Electromotive forces produced in a conductor in any one of these ways are said to be "induced" in the conductor, and are briefly described as "induced" electromotive forces. The

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phenomena described are usually referred to as electromagnetic phenomena, from the fact that they are intimately related to that property of a current in virtue of which a piece of iron, when placed in its vicinity, becomes a magnet (see Article 24).

**75. Magnetic Fields.**—The magnetization of a piece of iron when placed in the vicinity of a conductor carrying an electric current, and the electromagnetic phenomena above described, lead to the conclusion that an electric current establishes a peculiar state or condition at each point in the space surrounding its path. That is, the flow of electricity not only produces effects (such as heating and chemical action) *in* the path along which this flow takes place, but also produces effects in the region *surrounding* this path.

Since one of the effects produced by an electric current in the region surrounding its path is the magnetization of any piece of iron which may be in this region, the peculiar state or condition
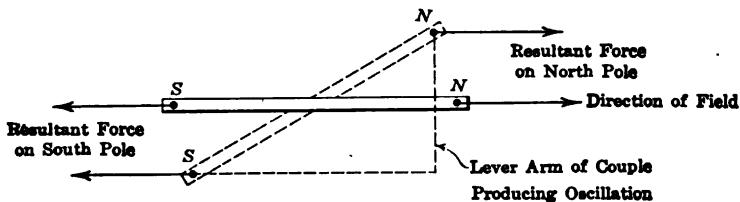


FIG. 38.

which exists in the region surrounding an electric current may be conveniently designated as a *magnetic* state, and the region itself may be called the "magnetic field" of the current.

Experiment shows that the effects produced in a magnetic field are of such a nature as to lead to the conclusion that the magnetic state at each point is characterized both by a magnitude and a direction, and that in general both the magnitude and direction of this state vary from point to point. In this respect the magnetic state in the region surrounding an electric current is analogous to the stress produced at the various points in a bent beam, or in any deformed structure.

The directive nature of the magnetic state at any point may be readily shown by placing at this point a small magnetic needle, suspended at its center from a string which produces negligible torsion. When the needle is so placed in the field, it will in general make a number of oscillations, but will ultimately come to

rest in a definite position, with its normally north-seeking end (or north "pole") pointing in a definite direction.

The oscillations of the needle, and the fact that it ultimately comes to rest in a definite position, indicate that the two ends (strictly the two halves) of the needle are acted upon by forces which are in opposite directions, the line of action of each force being parallel to the final position of the axis of the needle when it comes to rest (see Fig. 38). These forces may be looked upon as due to the magnetic state which exists at the point at which the needle is placed. This magnetic state, however, must not be thought of as existing only when the needle is present; it exists whether the needle is present or not. The needle is merely
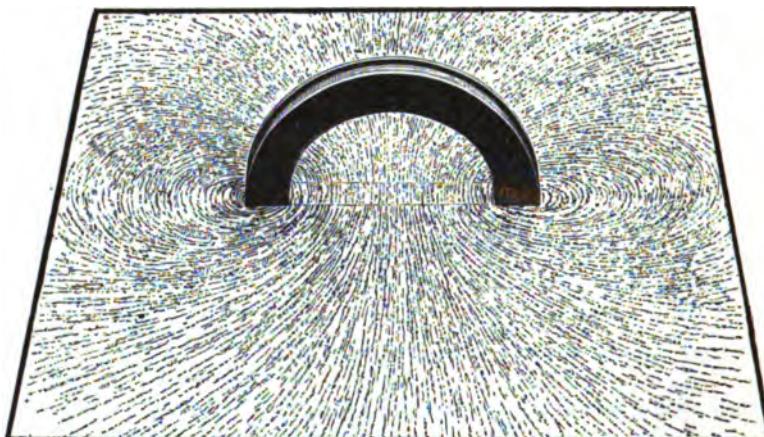


FIG. 39.—Lines of magnetic force due to current in a circular coil.

a means by which the magnetic state which exists at the given point may manifest itself in a visible effect.

The direction taken by a freely suspended magnetic needle, when placed at a given point, is arbitrarily taken as the direction of the magnetic state which exists at this point. This direction is usually referred to as the direction of the magnetic field at the given point. The positive sense of the field is the direction in which the normally north-seeking end, or north pole, of the test needle points.

This definition of the direction of a magnetic field at any point is mathematically exact only when it is further specified (1) that the test needle be infinitely small, (2) that it be placed at the given point without removing from the field any matter other

than that originally in the infinitesimal space which the needle is made to occupy, and (3) that no external force, having a component aiding or opposing the forces due to the field, is exerted upon the needle.

**76. Lines of Magnetic Force.**—Just as the direction of the flow of electricity in a conductor may be represented by "stream lines" drawn in the conductor (see Article 64), so may the direction of a magnetic field at its various points be represented by lines so drawn that their direction at each point coincides with the direction of the field at that point. Such lines are called "lines of magnetic force" or "lines of magnetic induction."
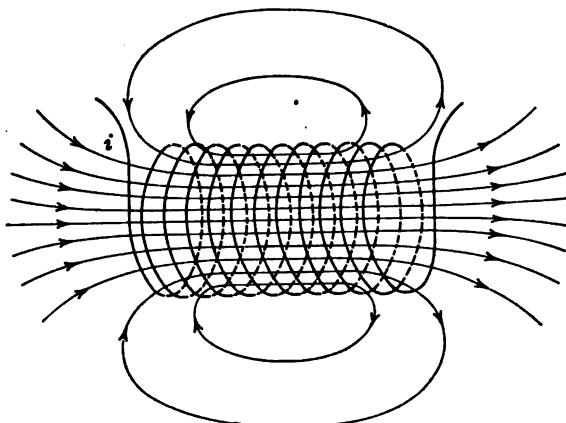


Fig. 40.—Lines of force due to a current in a solenoid. (Only the lines in the plane of the paper are shown. There are similar lines in every plane through the axis of the solenoid.)

The shape of the lines of force in any magnetic field may be roughly determined, in a very simple manner, by placing horizontally in the field a piece of cardboard, or other flat surface, and sprinkling fine iron filings on this surface. When the surface is tapped lightly, the filings will arrange themselves along definite lines, coinciding with the horizontal projection of the lines of force on the given surface. The explanation of this effect is that each filing becomes magnetized, *i.e.*, acquires the properties of a small magnetic needle, and when the surface is tapped, each of these little magnets tends to take up the same direction as would be taken by a magnetic needle freely suspended at the point where this particular filing is located.

In Fig. 39 is shown the distribution of iron filings, on a hori-

zontal surface, produced by a current in a circular coil perpendicular to and bisected by this surface. The alignment of these filings shows very clearly the shape of the lines of force due to the current in the coil.

In Fig. 40 are shown the lines of force (in one plane only) due to a current in a helical coil of circular cross-section. Such a coil is called a "solenoid." In Fig. 41 are shown the lines of force due to a permanently magnetized bar. (A piece of hard steel may be permanently magnetized by sending a current through a coil of insulated wire which is wrapped around the bar. When the bar is removed from the coil, it is found to retain its
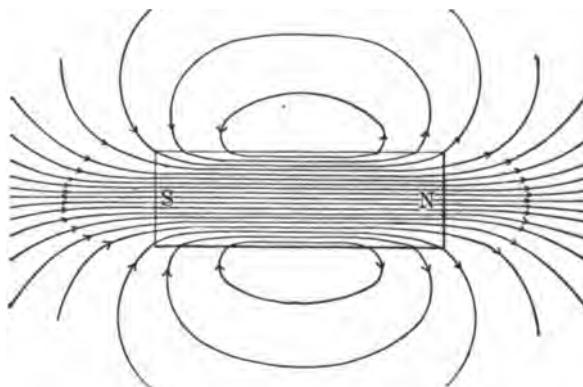


Fig. 41.—Lines of force due to a bar magnet. (Only the lines in the plane of the paper are shown. There are similar lines in every plane through the axis of the magnet.)

magnetic properties. This, in fact, is the usual way of making ordinary bar and "horseshoe" magnets.)

**77. Right-handed and Left-handed Screw Relations.**—An inspection of Fig. 40 will show that the relative direction of the lines of force through the coil and the direction of the current in the wire which forms the turns of the coil, is the same as the relative direction of the motion of the tip of a right-handed screw (see Fig. 42) and the direction of rotation of a point on the head of this screw, when the screw is turned in or out of a solid block. Any two quantities which have such relative directions are said to be in the *right-handed screw direction with respect to each other.*

Conversely, when the directions of two quantities are relatively the same as the direction of motion of the tip of a left-handed screw and the direction of rotation of a point on the head of this

screw, they are said to be in the *left-handed screw direction with respect to each other*.

**78. Continuity of the Lines of Magnetic Force.**—Experiment shows that every line of magnetic force, as above defined, is a closed loop; *i.e.*, a point which is moved continuously in the direction of the field traces a path which always closes on itself. An infinite number of such closed loops may be drawn in a magnetic field.

It is also an experimental fact that every line of force either **links the path of one or more electric currents (Fig. 40) or passes through a permanent magnet (Fig. 41)**, or both. Moreover, experiment shows that the lines of magnetic force due solely to a current in a *single* coil always link this current in the right-handed screw direction* and the magnetic lines of force due
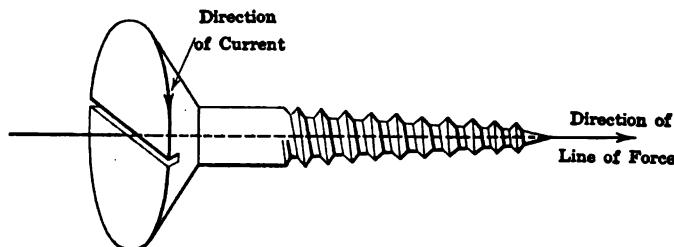


FIG. 42.

solely to a *single* permanent magnet always pass through the magnet from its south to its north pole.

The reader should note at this point that some writers use the term "lines of force" to designate a set of lines defined in such a manner that they are not necessarily closed loops, but in general end at the magnetic poles which may be in the field. The closed loops above described are then designated specifically as "lines of induction," and not as "lines of force." Unfortunately, however, this distinction is not recognized in common usage, but the term "lines of force" is used indiscriminately to designate *both* sets of lines.

This double use of the term "lines of force" often leads to much confusion, particularly on the part of the beginner. To

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* When a coil surrounds a magnetic core which has been previously magnetized, the resultant lines of force due to both the current and the magnetized state of the core may, under certain special conditions, link the current in the left-handed screw direction (see Article 111).

avoid this confusion, and at the same time to retain the term "lines of force" in its more common meaning, this term will be used throughout this book, when referring to magnetic fields, to designate only the *closed* loops above described. The other set of lines will be referred to as "lines of *magnetizing* force" (see Article 90) and *never* by either of the terms "lines of magnetic force" or "magnetic lines of force."

**79. Magnetic Flux.**—Just as an electric current can be expressed quantitatively only in terms of some measurable effect which it produces (see Article 28), so can the magnetic state in any region of a magnetic field be expressed only in terms of some effect which is caused by this state.

In the preceding articles it has been shown how the *direction* of the magnetic field at any point can be expressed in terms of the direction taken by a small magnetic needle freely suspended at that point. The direction taken by such a needle, as already noted, is due to forces which act in opposite directions on the two halves of the needle, which forces form a couple for all positions of the needle except that in which the axis of the needle coincides with the line of action of these forces.

As a measure of the *magnitude* of the magnetic state at any point in a magnetic field may be taken the force which would act on one end, say, the north-seeking end, of a given permanent magnetic needle, when freely suspended at the point in question. In fact, this method of expressing the "strength" of a magnetic field is the one originally adopted by physicists, and is still commonly employed. This method of procedure, however, invariably leads to confusion when applied to points *inside* a magnetic substance, such as iron. Consequently, a less ambiguous method of expressing the quantitative relations in a magnetic field will be adopted here.

The effect which will here be used, as the basis of the quantitative measure of a magnetic state or condition, is the electromotive force which is induced in a conductor by a varying magnetic field (see Article 74). In the first place, experiment shows that *during the establishment* of a magnetic field through a closed loop (see Fig. 43), an electromotive force is always induced in this loop. As already noted, an induced electromotive force exists only while the magnetic field is *changing*. When there is no magnetic field, there is no electromotive force, and when the

field becomes constant, the electromotive force is likewise zero; but during the time required to establish the field an electromotive force exists in the loop. Consequently, though the electromotive force is zero at the beginning and at end of the period required to establish the field, the *time integral*[1] of the electromotive force during this period is not zero.

As will be shown later, the time integral of an electromotive force can be readily determined experimentally by means of a ballistic galvanometer (see Article 86), in the same manner as *quantity* of electricity, which is the time integral of an electric *current*, is determined. By such measurements it may be shown that the time integral of the electromotive force induced in a loop of fixed size, shape and position, when a magnetic field is established through it, is independent of the rate at which the field is estab-
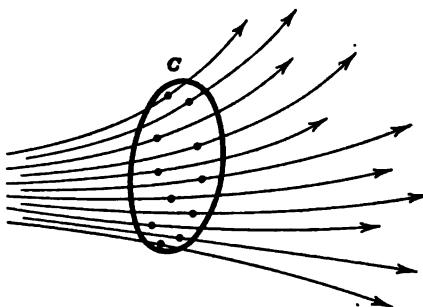


Fig. 43.

lished, but depends solely upon the *final magnetic condition* established through the surface bounded by this loop.

Consequently, as the measure of the magnetic condition at any surface in a magnetic field may be taken the time integral of the electromotive force which is induced, during the establishment of this field, in the closed loop formed by the boundary of this surface. The magnetic condition at a surface in a magnetic field is conveniently referred to as the "magnetic flux" through this surface, just as the flow of electricity through a surface is called the electric current through this surface. The symbol $\varphi$ is almost invariably used to designate a magnetic flux. Hence, the fundamental definition:

[1] By the time integral of a quantity is meant its average value multiplied by the interval of time in question.

As the quantitative measure of the magnetic flux ($\varphi$) through any surface is taken the time integral of the electromotive force ($e$) which is induced, during the establishment of this flux, in the closed loop formed by the boundary of this surface, viz.,

$$\varphi = \int_0^t e\,dt \qquad (1)$$

The use of the term "flux" for the quantity here defined arises from the fact that this quantity is found to have many properties analogous to those of an electric current, or "flow" of electricity. However, there is one important difference between a magnetic flux and an electric current, which the student should never lose sight of, viz., an electric current is an actual flow of something, but a magnetic flux is a state or condition.

This method of specifying the magnitude of the magnetic flux through a surface, though at first sight it may seem complicated, is inherently no more complex than the definition, for example, of acceleration as the *time-rate of change* of velocity. The conception of an integral with respect to time is no more complex than that of a rate of change (or derivative) with respect to time. Moreover, the definition expressed by equation (1) involves the particular phenomena of magnetism which is of the greatest practical importance, namely, the phenomenon of electromagnetic induction. In short, this definition is nothing more than a brief statement of the method usually employed for measuring the flux through a given surface (see Article 86).

The fundamental point of similarity between a magnetic flux and an electric current is that the magnitude and direction of the magnetic flux in the various parts of a magnetic field may be represented by lines, just as the distribution of current in a conductor may be represented by lines of flow, or stream lines.

Consider in the field any surface (Fig. 44) which is perpendicular at each point to the line of magnetic force through that point, and which intersects every line of force which can be drawn in the field.[1] Such a surface is called a "magnetic equipotential surface." Imagine this surface to be divided into small areas such that the magnetic flux, as defined by equation (1), has the value unity for each of these areas; i.e., each of these areas is such that the time integral of the electromotive

[1] Only a portion of such a surface is shown in Fig. 44.

force induced in the closed loop formed by its boundary, when the magnetic field is established, is equal to unity.

Through the center of each of these areas draw a line of force, and extend these lines until each closes on itself. Then the number of these lines which intersect *any surface whatever* in the field, irrespective of the size, shape or position of this surface, is equal to the magnetic flux through this surface; *i.e.*, is equal to the time integral of the electromotive force induced in the closed loop formed by its boundary during the establishment of the field. This relation, of course, does not follow as a necessary consequence of the definition of magnetic flux; it is an *experimental fact*.

The magnetic flux through a loop is therefore commonly spoken of as the number of lines of force (or number of lines of induction)



Lines of Force

Magnetic Equipotential Surface
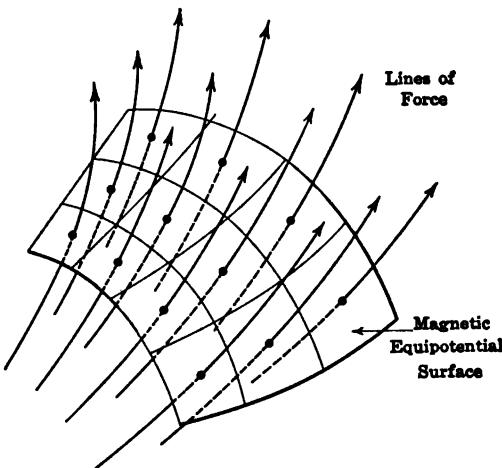
Fig. 44.

which thread, or link, this loop. This, however, is not a definition of *magnetic flux*, but, conversely, is a definition of the *number* of lines of force, for up to the present the term "line of force" has been used merely to signify the *direction* of the magnetic field at a given point. An infinite number of such direction-lines can be drawn in a given field. However, when the lines of force in a given field are limited to a finite number, in the manner above described, the term "number of lines of force" becomes a very convenient one for designating the magnetic flux. When the number of lines of force in a magnetic field is

limited, in the manner just described, these lines may be conveniently referred to as the "lines of magnetic flux," or "flux lines."

It should be noted that a single line of force, drawn as above described, actually represents a *tube* whose walls are everywhere tangent to the direction of the field, and through each cross-section of which there exists unit flux. Hence a line of force, though usually represented by a geometrical line, must be considered as having a definite cross-section. A line of force is, therefore, analogous to a wire bent to form a closed loop, except that the cross-section of the line of force is not, as a rule, uniform (see Fig. 49).

**80. Units of Magnetic Flux.**—From equation (1) it follows that, in any infinitesimal interval of time $dt$, the change of the magnetic flux which threads a given loop is

$$d\phi = edt$$

where $e$ is the electromotive force induced in this loop during this interval. From this relation, it is evident that magnetic flux has the same "dimensions" as the product of electromotive force by time, just as energy has the same "dimensions" as the product of power by time. Hence, just as the fundamental practical unit of energy may be called the watt-second, so may the fundamental practical unit of flux be called the **volt-second**.

Unfortunately, this practical unit of magnetic flux is not in general use. Instead, the corresponding unit in the c.g.s. electromagnetic system is generally employed, even in engineering calculations. This latter unit, which is the product of an abvolt (1 abvolt = $10^{-8}$ volts) by a second has been given the name "maxwell." The relation between the maxwell and the volt-second is then:

$$1 \text{ volt-second} = 10^8 \text{ maxwells}$$

When the magnetic flux $\phi$ through a loop is expressed in maxwells, time $t$ in seconds, and the induced electromotive force $e$ in volts, equation (1) becomes

$$\varphi = 10^8 \int_o^t edt \tag{1a}$$

A maxwell is also frequently called a c.g.s. "line." That is, by a flux of 5,000,000 lines is meant a flux of 5,000,000 maxwells, which in turn is equivalent to a flux of 0.05 volt-seconds. The

term "kiloline" is used to designate a flux of 1000 lines or maxwells.

**Problem 1.**—A given closed loop of wire has a resistance of 3 ohms. (a) If at any instant the electromotive force induced in this loop is $e$ volts, what will be the current in this loop at this instant? (b) What will be the quantity of electricity transferred during an interval of $dt$ seconds through each cross-section of the wire which forms this loop? (c) If during the establishment of a magnetic field through this loop the total quantity of electricity transferred through each cross-section of this wire is 0.002 coulombs, what will be the magnetic flux established through the loop, in volt-seconds? (d) In maxwells?

**Answer.**—(a) $\frac{e}{3}$ amperes.  (b) $\frac{edt}{3}$ coulombs.  (c) 0.006 volt-seconds.
(d) 600,000 maxwells.

**Problem 2.**—(a) From equation (1a), what is the relation between the electromotive force, in volts, induced at any instant in a loop of wire and the variation, with respect to time, of the magnetic flux, in maxwells, which threads this loop. (b) If during an interval of 0.001 second, the flux which threads the loop changes from 200,000 to 500,000 lines, what will be the average electromotive force induced in the loop during this interval?

**Answer.**—(a) The induced electromotive force in volts is equal to $10^{-8}$ times the change per second in the number of lines of force which thread the loop, or

$$e = 10^{-8}\frac{d\varphi}{dt} \qquad \text{volts}$$

(b) 3 volts.

## 81. Induced Electromotive Force and Rate of Change of Flux.

—The value of the conception of magnetic flux, or *number* of lines of force, is that it leads to a simple method of stating, in a *quantitative* manner, the various phenomena of electromagnetic induction described at the beginning of this chapter, namely:

(A) When any portion of an electric circuit moves with respect to the agents which produce a magnetic flux, the electromotive force induced in this portion of the circuit is always equal to the time rate at which this portion of the circuit cuts the lines of force which represent this flux.

(B) When the number of lines of force through any area of fixed size and shape changes in any manner whatever, there is always induced in the boundary of this area an electromotive force equal to the time rate of change of the number of lines of force which thread this area, and, therefore, link its boundary (see also Article 85). The area in question may be either in a medium which is fixed with respect to the agents which produce the flux, which flux may vary with time; or it may be in a substance which

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is in motion with respect to the agents which produce the flux which flux may either be constant or vary with respect to time.

In either case, the induced electromotive force may be expressed by the formula

$$e = \frac{d\phi}{dt} \qquad \text{abvolts} \qquad (2)$$

In the first case, $e$ is the electromotive force induced at any instant in the portion of the circuit which at this instant is cutting lines of force at the rate of $d\phi$ lines in $dt$ seconds. In the second case $e$ is the total electromotive force induced in a closed loop of given size and shape when the number of lines of force which link this loop is changing at the rate of $d\phi$ lines in $dt$ seconds.

When the change in the number of lines of force which thread a closed loop is due to the motion of this closed loop with respect to the agents which produce the field, it is immaterial whether the induced electromotive force be looked upon as due to the cutting of lines of force by the conductor which forms the loop,
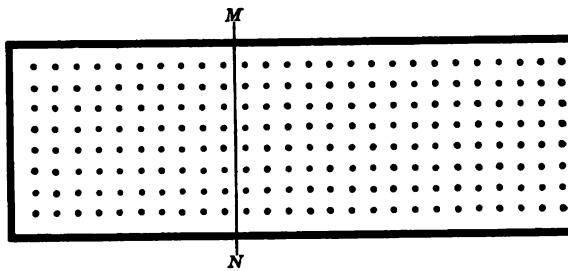


FIG. 45.

or to a change in the number of links, or "linkages," between the loop and the lines of force which represent the flux.

By making the assumption that the lines of force produced by a current in a wire all "originate" in the axis of the wire and spread out from the wire when the current increases, and collapse on the wire when the current decreases, every induced electromotive force may be looked upon as due to the cutting of lines of force by the conductor in which the electromotive force is induced. However, when the two possible interpretations of $\frac{d\phi}{dt}$ in equation (2) just noted are kept in mind, it is unnecessary to make any such assumption.

**Problem 3.**—In Fig. 45, the heavy line represents a wire, and *MN* represents a slider which may move parallel to itself over the two "rails" formed by the wire. This electric circuit is in a magnetic field, the lines of force of which are perpendicular to the plane formed by the wire and the slider, and the magnetic flux through each square inch of the plane in which this loop is located is *B* maxwells. The length of the slider between the two rails is *l* inches.

(*a*) If the slider is moving with a velocity of *v* inches per second what will be the change per second in the number of lines of force which thread the loop of which the slider forms one side? (*b*) What will be the rate at which the slider cuts the lines of force? (*c*) What will be the electromotive force induced in this loop? (*d*) From the point of view of the cutting of lines of force, in what part of the loop is this electromotive force induced?

*Answer.*—(*a*) *Blv* maxwells per second. (*b*) *Blv* maxwells per second.

(*c*) $$e = Blv \qquad \text{abvolts.} \tag{2a}$$

(*d*) In the slider.

## 82. Direction of the Induced Electromotive Force.—Resultant Flux Through a Surface.

—Experiment shows that the direction of the electromotive force which is induced in any loop, by a change in the magnetic flux which threads it, is always such that it would, if acting alone, set up a current which would of itself produce a magnetic field in such a direction as to oppose this change. This relation is known as "Lenz's Law."

As noted in Article 77, the lines of force produced by a current in a conducting loop always thread this loop in the right-handed screw direction. Hence, an **increase** in the number of flux lines which thread a loop always induces in the loop an electromotive force in the **left-handed screw direction** with respect to the direction of the flux lines. Similarly, a **decrease** in the number of flux lines which thread a loop always induces in the loop an electromotive force in the **right-handed screw direction** with respect to the direction of the flux lines.

For example, consider a loop of wire lying on a table, and assume that there are originally no lines of force threading it. Now bring up a magnet, or a coil carrying a current, so that, say, $\varphi$ lines of force thread this loop in the direction vertically downward. The electromotive force induced in the loop while the flux threading it increases from zero to $\varphi$ will then be around the loop in the counter-clockwise direction as viewed from above. If the flux threading the loop is now decreased from $\varphi$ to zero, by removing the agent which produces the magnetic field, the

electromotive force induced in the loop, while the flux is decreasing, will be in the clockwise direction as viewed from above.

Again, if the agent which produces the field is so placed that the lines of force due to it pass vertically upward through the loop, then during the establishment of the flux linkages between the loop and the field, the induced electromotive force will be in the clockwise direction around the loop as viewed from above. If the agent which produces the field is now removed, then during the decrease in the flux linkages between the loop and the field, the induced electromotive force will be in the counterclockwise direction around the loop as viewed from above.

A flux which threads a coil in a given direction, say, from $A$ to $B$, therefore induces, when it is established or when it disappears, an equal and opposite electromotive force to that which is produced by the establishment or disappearance of an equal flux which threads this coil in the opposite direction, *i.e.*, from $B$ to $A$. Hence, when a coil is threaded by, say, $\varphi_1$ lines in the direction from $A$ to $B$, and $\varphi_2$ lines in the direction from $B$ to $A$, it may be considered as threaded by $(\varphi_1 - \varphi_2)$ lines of force in the direction from $A$ to $B$, or by $(\varphi_2 - \varphi_1)$ lines in the direction from $B$ to $A$.

That is, magnetic fluxes are to be treated as algebraic quantities, and added and substracted algebraically. Any chosen direction through a loop or surface may be taken as the positive sense of the flux. All fluxes which thread this loop or surface in this chosen direction are then to be considered as positive, and all fluxes which thread this loop or surface in the opposite direction are to be considered as negative.

For example, consider a circular coil which is threaded by $\varphi$ flux lines in the direction, say, from its $A$ face to its $B$ face. Let this coil be turned 180 degrees about any diameter; it will then be threaded by $\varphi$ flux lines in the direction from its $B$ face to its $A$ face. The change in the flux which threads the coil, due to its being turned through 180 degrees, is then $2\varphi$.

Again, when the flux through a stationary coil is changed from the value $\varphi$ in one direction to an equal value in the opposite direction, the total change in the flux through the coil is likewise $2\varphi$.

When the induced electromotive force is due to the cutting of the lines of force by a moving conductor, as illustrated in

Problem 3, Lenz's Law may be stated in another form, which is often very convenient. Referring to Fig. 45, assume that the lines of force which thread the loop are in the direction into the plane of the paper away from the eye of the reader, and let the slider *MN* be moving toward the right. The flux which threads the loop in the direction away from the eye of the reader is then increasing, and, therefore, from Lenz's Law, the electromotive force induced in the loop will be counter-clockwise, *i.e.*, the electromotive force in slider will be in the direction from *N* to *M*. This direction is the same as that in which the middle finger of the right hand points when the forefinger is pointed in the direction of the lines of force and the thumb in the direction of the motion of the slider.

This relation is a perfectly general one, *i.e.*, the electromotive force induced in a wire, due to its motion in a magnetic field, is always in the direction in which the **middle finger** of the **right hand** points when this finger is placed along the wire and the **forefinger** is pointed in the direction of the lines of **force** and the **thumb** in the direction of the **motion** of the conductor. (In applying this rule the thumb and forefinger must be kept parallel to the palm of the hand.) This rule is commonly known as the "right-hand rule." It must not be confused with the right-handed screw rule for the relation between a current and the lines of force due to this current.

Problem 4.—What is the direction, with respect to the current in a coil, of the electromotive force induced in this coil by the magnetic field due to this current, (*a*) when the current is increasing? (*b*) when the current is decreasing?

*Answer.*—(*a*) When the current is increasing, the induced electromotive force due to the increase in its own magnetic field opposes the current, *i.e.*, is a *back* electromotive force. (*b*) When the current is decreasing, the induced electromotive force due to the decrease in its own field is in the same direction as the current.

**83. The Direct-current Dynamo.**—The production of an electromotive force in a conductor which is moved in a magnetic field in such a manner that it cuts across the lines of force which represent this field, is the fundamental principle involved in the operation of the direct-current dynamo. Such a machine may be used either as a generator of electric energy, when driven by a "prime mover," *e.g.*, a steam engine or water-wheel; or it will operate as an electric motor, when supplied with electric

energy from a direct-current generator or other source of direct current. Alternating-current generators and motors also operate on this same principle (see Article 128).

In a direct-current dynamo (usually abbreviated d.c. dynamo) the conductors in which the electromotive force is induced are usually insulated copper wires or copper bars, called "armature conductors," which are imbedded in slots in the exterior surface of a hollow iron cylinder (see Fig. 46). These slots are parallel to the axis of the cylinder. The cylinder itself, which is called
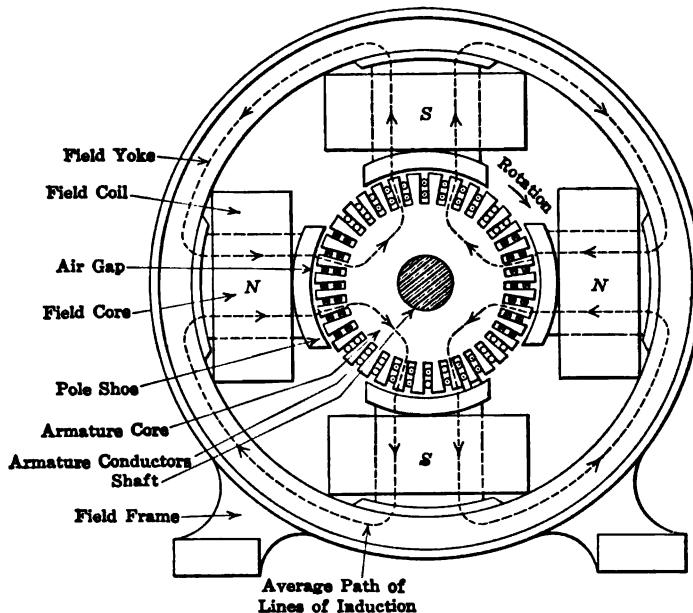


FIG. 46.—D-C. Generator. Cross-section perpendicular to shaft.

the "armature core," is made up of sheets of soft iron or steel; the planes of these sheets are perpendicular to the axis of the cylinder. In large machines these flat rings of sheet steel are fitted on to a cast-iron frame, called the "armature spider," which resembles the spokes and hub of a wheel. The armature spider (or, in small machines, the steel sheets themselves) is mounted on a shaft or axle which runs through it and projects out at each end. The ends of the shaft are mounted in suitable bearings, and, in case the machine is to be driven by, or is to drive, a belt, a pulley is mounted on one end of the shaft. The armature conductors,

core and spider taken collectively are called the "armature" of the machine.

The magnetic field in which the armature conductors are rotated is produced by an electric current in two or more coils of insulated wire, called "field coils," which are wound on stationary iron or steel cores placed symmetrically around the armature core, as shown in the figure. The ends of these "field cores" next the armature are broadened out, so that they cover from 50 to 70 per cent. of the armature surface, and are made concave toward the armature so that the end surfaces, called the "pole faces," form part of a cylindrical surface concentric with the armature and of a slightly greater radius. These broadened ends of the field cores are called the "poles shoes," and are frequently made separate from the field cores and are bolted to the latter when the machine is assembled. The ends of the field cores away from the armature are connected by a yoke of iron or steel; this yoke is called the "field yoke." The field coils, field cores, pole shoes and field yoke taken collectively are called the "field" of the machine.

The air space between the pole shoes and the armature is called the "air gap." The lines of force produced by the current in the field coils form closed loops, such as indicated by the dotted lines in the figure. Those field poles from which the lines of force pass out into the air gap are called the "north poles" of the machine, and those into which the lines of force enter from the air gap are called "south ploes" (see Article 103).

In the simplest type of armature winding, the armature conductors are all connected in series by insulated copper wires or bars, called "end connections," across the two ends of the armature core, so that the armature winding forms a **closed** coil around this core. The net electromotive force induced in this closed winding is the sum of the electromotive forces induced in the individual armature conductors. The field winding produces practically no lines of force which cut the end connectors; the entire electromotive force induced in the armature winding is that due to the cutting of lines of force by the armature conductors which lie in the air gap.

Hence, when the armature conductors are distributed symmetrically around the armature core, the net electromotive force induced in the armature winding is zero, since for each

conductor cutting the magnetic field on one side of the armature, there is a corresponding conductor cutting an equal field on the other side of the armature, in the opposite direction with respect to the motion of the conductor. A study of Fig. 46 will make this clear; a dot in an armature conductor indicates an electromotive force in the direction toward the reader, and a cross an electromotive force in the direction away from the reader. The electromotive force induced in those conductors not under a field pole is practically zero.

However, although the net electromotive force induced in the complete armature winding is zero, a constant difference of electric potential can be obtained from the armature winding by
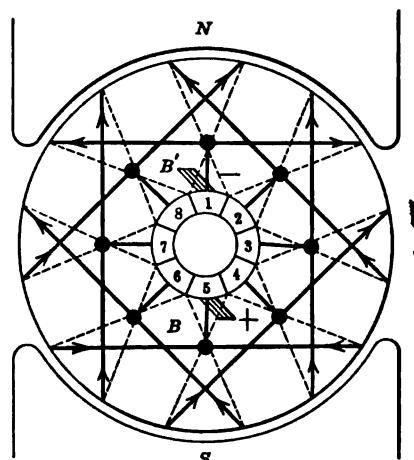


FIG. 47.—Simple armature winding.

bringing out suitable connections, or "taps," and making contact with these connections in a suitable manner. This can be best understood from Fig. 47, which shows diagrammatically an armature winding of a simple two-pole machine, with the end connections on the end of the armature facing the reader as heavy lines, and the end connections on the other end as dotted lines.

The middle point of each of the front connections is connected to a bar of a device called a "commutator," which is a cylinder of copper bars insulated from each other by sheets of mica. This cylinder is mounted rigidly on the shaft of the armature, from which the bars are also insulated. From the segment marked

1 there are two paths through the armature winding to the segment marked 5, and the electromotive force induced in each of the armature conductors in each of these paths is in the direction from 1 to 5. Hence the net electromotive force between 1 and 5 through each of these paths is the arithmetical sum of the electromotive forces induced in all the conductors in this path, *i.e.*, in half the total number of armature conductors.

The two halves of the armature winding are then similar to two electric batteries in parallel. The electromotive force of one-half of the winding opposes the electromotive force induced in the other half of the winding, and consequently no current flows in the closed circuit formed by the entire winding. However, when the segments 1 and 5 are connected to an external circuit, the electromotive force impressed on this circuit will be equal to the electromotive force induced in each half of the winding, less the internal resistance drop, and consequently a current will be established in the external conductor, one-half of the current flowing through each half of the armature winding.

As the armature turns in the direction indicated, the electromotive forces induced in the individual conductors which form each path between 1 and 5 will not remain in the same direction, but some of the electromotive forces will reverse with respect to the others, and hence the net electromotive force between 1 and 5 will decrease. However, when the armature has rotated through an angle corresponding to one segment of the commutator, another pair of segments, 8 and 4, come into the position formerly occupied by 1 and 5, and the electromotive force between 8 and 4 will be exactly the same as the electromotive force which previously existed between 1 and 5, provided the armature rotates with a constant speed. Similarly for the next pair of segments, and so on.

Consequently, if, in the positions *B* and *B'* are mounted two fixed contacts, under which the commutator segments slide as the armature rotates, the electromotive force between these two contacts will remain practically constant. The greater the number of commutator segments, the more nearly constant will this electromotive force be. The fixed contacts which rub against the commutator segments are called "brushes," and in most modern machines are made of carbon blocks. These brushes are held in a suitable support, called the "brush-holder." The

parts of this brush-holder in contact with the brushes are insulated by suitable bushings from the brackets which connect them to a common support, called the "rocker-arm," which is mounted on some part of the stationary structure which forms the frame of the machine. This rocker-arm is so mounted that the position of the brushes can be adjusted until no sparking occurs between them and the commutator segments when a current is taken from or supplied to the machine.

Dynamos designed for the generation or utilization of large amounts of electric power always have more than two poles, and in most cases a corresponding number of sets of brushes. In such "multipolar" machines all the positive brushes are interconnected and all the negative brushes are interconnected. The armature in such a multipolar machine may have a single winding of the form just described, or there may be two or more independent windings on the same armature core, connected in parallel.

The several field coils of a dynamo are always connected in series electrically, and this complete field circuit may be connected either in shunt or in series with the armature, as illustrated in Figs. 21 (p. 78) and 27 (p. 97). When the field circuit is shunted around the armature, the machine is called a "shunt" machine, and when the field circuit is in series with the armature the machine is called a "series" machine. Frequently two sets of field coils are employed, one set in shunt and the other in series with the armature or with the external load supplied by the machine. A machine which has both series and shunt field coils is called a "compound" machine. The field current may, of course, be supplied by a source of electromotive force which has no connection with the armature circuit; the machine is then said to be "separately excited."

A self-excited machine (shunt, series or compound) when started up as a generator will develop in its armature a small electromotive force, due to the residual magnetism in the field poles (see Article 111). This electromotive force in turn establishes a small current in the field windings, which increases the magnetic flux in the gap; this in turn increases the induced electromotive force, and this cumulative process goes on until the field current reaches a steady value. In the simple shunt-connected generator, for example, the field current increases until

the induced electromotive force in the armature establishes a difference of potential between the brushes of the machine equal to the product of the resistance of the shunt field by the strength of the current in this field.

The total current taken from the brushes of any type of continuous-current generator for a given induced, or "armature," electromotive force, will depend upon the resistances and electromotive forces in the external circuit, and also upon the internal resistance between the brushes and upon the resistances of the field windings. When the various resistances and electromotive forces are known, the strengths of the currents in the external circuit and the various windings of the machine can be calculated by Kirchhoff's Laws. Since in an electric generator the current always gains electric energy, the direction of the current through the armature is the same as that of the electromotive force developed in the armature. Hence the current always leaves the armature of a generator at the positive brush, and enters at the negative brush.

A direct-current dynamo (series, shunt or compound) may also be used as a motor. In this case, its terminals are connected to some source of potential difference which establishes a current through its field coils and through its armature. The magnetic field produced by the current in the field coils exerts a mechanical force on the armature and causes it to rotate (see Chapter XI). Mechanical work is then done on whatever is connected to the armature shaft, and the energy to do this work comes from the source of electromotive force (*e.g.*, the generator) which establishes the current through the machine. Since in a motor electric energy is always converted into some other form (*i.e.*, mechanical energy), the electromotive force developed by a motor is always in the direction opposite to that of the current; that is, *an electric motor always develops a back electromotive force*. The current, therefore, always enters the armature of a motor at the positive brush, and leaves it at the negative brush.

For a fuller description of the construction of electric dynamos and a discussion of the various factors which affect their operation as generators or motors, the reader is referred to any text-book on dynamo-electric machinery.

**84. Electromotive Force Induced in the Armature of a Direct-current Dynamo.**—An important practical application of equa-

tion (2) is the calculation of the value of the electromotive force induced in the armature winding of a direct-current dynamo.   Let

$Z$ = the total number of armature conductors.

$p$ = the number of field poles.

$a$ = the number of parallel conducting paths between the positive and negative brush sets, that is, $\dfrac{Z}{a}$ is equal to the number of armature conductors in series between the positive and negative brush sets.

$\phi$ = the total useful magnetic flux per pole in c.g.s. lines or maxwells; that is, $\phi$ is the total number of lines of force which pass *through the armature core* from a north pole to the adjacent south pole of the field.

$n$ = the number of revolutions of the armature per minute.

The time required for each conductor to pass entirely around the armature is then $\dfrac{60}{n}$ seconds, and, therefore, the time taken for it to pass through the magnetic field under each pole is $\dfrac{60}{np}$ seconds.   Hence the average value of the electromotive force induced in each armature conductor as it passes under each pole is $\dfrac{np\varphi}{60 \times 10^8}$ volts.   Since the conductors are uniformly distributed around the surface of the armature, this is also the average value at each instant of the electromotive force induced *per conductor* in each of the parallel paths between the positive and negative brushes.   Since there are $\dfrac{Z}{a}$ conductors in series between the positive and negative brush sets, the average value of the *total* electromotive force between the brushes is

$$E = \frac{10^{-8}}{60} \cdot \frac{pZ}{a} \cdot n\varphi \qquad \text{volts} \qquad (3)$$

This electromotive force is usually referred to as the "armature" electromotive force, to distinguish it from the terminal voltage, which is less than this electromotive force by the internal resistance drop (see equation (6), Chapter II).

When the number of commutator segments is large, the value, at any particular instant, of the total electromotive force between the brushes is practically equal to this average value.   For, since the brushes always make contact with segments in the *same*

*position with respect to the field* which produces the flux, the only possible variation in the electromotive force between the brushes (when $\phi$ and $n$ remain constant) is the variation which occurs as the armature rotates through an angle corresponding to one commutator segment. But, during the time required for this small displacement of an armature conductor, the *rate* at which each armature conductor cuts the lines of force remains practically constant (except for the three or four conductors which are just passing under, or are just leaving, a pole tip). Therefore, the total induced electromotive force, which is the *sum* of the electromotive forces induced in $\dfrac{Z}{a}$ conductors, does not change appreciably in this interval, provided $\dfrac{Z}{a}$ is a large number.

Equation (3) holds whether the dynamo is used as a generator or as a motor. In the case of a generator, this electromotive force is in the direction of the current, and in case of a motor in the opposite direction from that of the current. It should be noted, however, that the flux per pole in a generator or motor in general depends not only upon the current in the field coils, but also to a minor, but appreciable, extent upon the current in the armature. The armature current, under ordinary operating conditions, tends to produce a flux which opposes that due to the current in the field coils. This effect is known as "armature reaction."

**Problem 5.**—A certain 250-kilowatt direct-current generator has 8 poles, 768 armature conductors, 8 parallel paths between brush sets, and the flux per pole due to the field current alone is $13.5 \times 10^6$ lines. The generator is driven at a speed of 150 revolutions per minute. The total resistance of the armature winding (the 8 paths in parallel) is 0.007 ohm. Between each set of brushes and the commutator there is a "brush contact drop" of 1 volt (due to the resistance of these contacts). When the armature is delivering 250 kilowatts to the brushes the flux per pole drops to $13.0 \times 10^6$ lines, due to the demagnetizing action of the armature current.

(a) What is the armature electromotive force at no load? (b) At full load of 250 kilowatts? (c) What is the voltage between the positive and negative brushes at this load?

*Answer.*—(a) 259 volts. (b) 250 volts. (c) 241 volts.

**85. Flux Linkages.**—As shown in Article 81, the electromotive force induced in a *loop of a single turn*, due to the variation in the magnetic field threading this loop, is equal to the time rate of change of this flux, viz.,

$$e = \frac{d\varphi}{dt} \qquad \text{abvolts}$$

Experiment shows that, with respect to the electromotive force induced in it, a coil of $N$ turns is equivalent to $N$ single loops connected in series. When the $N$ turns are so close together that the same flux $\varphi$ links each turn, the electromotive force induced in each turn is $\frac{d\varphi}{dt}$ and, therefore, the total electromotive force induced in the coil of $N$ turns is

$$e = N\frac{d\varphi}{dt} \qquad \text{abvolts} \qquad (4)$$

or

$$e = 10^{-8}N\frac{d\varphi}{dt} \qquad \text{volts} \qquad (4a)$$

where $\varphi$ is in maxwells and $t$ in seconds, in both equations.

The product $N\varphi$, which represents the number of links between the wire and the lines of force which thread the coil, is called the number of "flux linkages" between the coil and the magnetic field. The symbol $\lambda$ will be used throughout this book to designate the flux linkages between an electric circuit and the magnetic field which threads this circuit.

In the particular case of a **concentrated** winding, *i.e.*, of a coil in which the turns are so close together that each of its $N$ turns may be considered as linking the same flux $\varphi$, the flux linkages are

$$\lambda = N\varphi \qquad (5)$$

In the case of a **distributed** winding, *i.e.*, a winding in which the several turns are linked by a different number of lines of force (see Fig. 40), the flux linkages between the winding and the magnetic field are

$$\lambda = \varphi_1 + \varphi_2 + \varphi_3 + \text{etc.} \qquad (5a)$$

where $\varphi_1$, $\varphi_2$, $\varphi_3$, etc., are respectively the fluxes (*i.e.*, the number of lines of force) which link the several turns which make up the winding.

The general expression for the electromotive force induced in any electric circuit by a change in the flux which links it is then

$$e = \frac{d\lambda}{dt} \qquad \text{abvolts} \qquad (6)$$

or

$$e = 10^{-8}\frac{d\lambda}{dt} \qquad \text{volts} \qquad (6a)$$

where λ represents the number of flux linkages between this circuit and the magnetic field which threads it.

**Problem 6.**—How many linkages are there between the solenoid and the lines of force shown in Fig. 40?

*Answer.*—135 linkages.

**86. Measurement of Magnetic Flux.**—From the definition of the quantitative measure of magnetic flux given in Article 79, the value of the flux through any portion of a magnetic field may readily be determined experimentally. The procedure is as follows:

Place in the field a coil of such a size and shape that it will be threaded by the flux to be measured. This coil may have any number of turns, say $N$, provided the turns are so close together that each is threaded by the same number of lines of force. Connect the two terminals of this coil, by means of a pair of insulated wires twisted together, to a "ballistic" galvanometer, *i.e.*, a galvanometer whose moving element has a relatively long period of oscillation (*i.e.*, several seconds).

When the moving element of the galvanometer has come to rest, suddenly remove the coil entirely from the field, *i.e.*, to a place at which there is no appreciable magnetic flux. The first swing of the moving element of the galvanometer is then a measure of the flux which linked the coil in its original position.

To prove this, note first that if $\varphi$ represents this flux, in maxwells, and $r$ is the total resistance, in ohms, of the test coil, leads and galvanometer, then from equation (4a),

$$\varphi = \frac{10^8}{N} \int_o^t e\,dt$$

where $e$ is the electromotive force in volts induced in the coil at any instant during its removal from the field, and $t$ is the time required to remove the coil from the field. At the instant at which the electromotive force is $e$, the current in the circuit formed by the coil, leads and galvanometer is $i = \frac{e}{r}$; whence $e = ri$. Therefore,

$$\varphi = \frac{10^8 r}{N} \int_o^t i\,dt$$

But $\int_o^t i\,dt$ is the quantity of electricity, in coulombs, which the induced electromotive force causes to flow through each section

of the circuit of which the coil is a part (see Article 28). Call this quantity of electricity $Q$. Then

$$\varphi = \frac{10^8 r}{N} Q \qquad \text{maxwells} \qquad (7)$$

It may be shown that when a given quantity of electricity $Q$ is suddenly discharged through a ballistic galvanometer, the first swing $D$ of its moving element is proportional to this quantity of electricity, provided the discharge takes place in an interval of time small in comparison with the period of the galvanometer. Hence the relation expressed by equation (7) may be written

$$\varphi = \frac{K}{N} D \qquad (7a)$$

where $D$ is the first swing of the moving element of the galvanometer when the test coil is removed from the field, $K$ a factor which depends on the characteristics of the particular galvanometer employed and the total resistance of the test circuit, and $N$ is the number of turns in the test coil. The factor $K$ may be readily determined experimentally by means of a standard solenoid (see Article 110).

Another, and often preferable, method of measuring the flux through a given region in a magnetic field, is to use a coil in series with a ballistic galvanometer as above described, but instead of pulling the coil out of the field, to turn it through 180 degrees about an axis through the plane of the coil. In this case, when the coil is symmetrical about the axis around which it is turned, the relation between the flux and the quantity of electricity transferred through it by the induced electromotive force is

$$\varphi = \frac{10^8 r}{2N} Q \qquad \text{maxwells} \qquad (7b)$$

The factor 2 arises from the fact that the total change in the flux which links the coil, from its $A$ face to its $B$ face is $2\varphi$ (see Article 82).

Again, when the flux is due to an electric current which may be interrupted or reversed, the test coil may be kept stationary, and the flux through it be decreased from $\phi$ to 0 or from $\phi$ to $-\phi$, by interrupting or reversing the current respectively.

**Problem 7.**—The calibration curve of a certain ballistic galvanometer is found to be a straight line having the equation $Q = 0.107D$, where $Q$ is

quantity of electricity, in microcoulombs, and $D$ is the first swing of the moving element in millimeters, as read by a telescope and scale. A rectangular coil of 20 turns, having a mean length of 10 inches and a mean width of ⅛ inch, is placed in the air gap of a dynamo, directly under the center of one of the pole faces, with its length parallel to the shaft of the armature, and with its plane tangent to the surface of the armature (so that the lines of force through the coil are perpendicular to the plane of the coil). The coil is connected in series with the ballistic galvanometer, whose resistance is 876 ohms. The resistance of the coil and the leads from it to the galvanometer is negligible in comparison with this resistance. When the coil is suddenly pulled out of the air gap, the first swing of the galvanometer is 146 millimeters. What is the flux in the air gap through the area (10 by ⅛ inches) originally occupied by the test coil?

*Answer.*—68,400 maxwells.

**87. Magnetic Flux Density.**—Just as the distribution of current in a conductor is conveniently expressed in terms of the current density at the various points of the conductor, so may the distribution of the magnetic flux in a magnetic field be conveniently expressed in terms of the "density" of the magnetic flux at the various points in the field. By the magnetic flux density at any point is meant **the flux per unit area through an element of a surface drawn normal to the direction of the field at this point.** That is, when the flux through such an area $dS$ is $d\varphi$, the magnetic flux density at the point at which $dS$ is taken is

$$B = \frac{d\varphi}{dS} \qquad (8)$$

When the flux density has the same value at every point of a finite area $S$, normal to the direction of the lines of force through it, equation (8) becomes

$$B = \frac{\varphi}{S} \qquad (8a)$$

This expression always gives the *average* density of a given flux $\varphi$ over an area $S$ normal to the lines of force, whether the flux density is constant or varies from point to point. Hence, since the flux through a given area is equal to the number of lines of force through this area (when the lines of force are drawn as specified in Article 79), the *average* flux density over any *unit* area, perpendicular to the lines of force which represent the field, is always equal to the number of lines of force which intersect this unit area.

Magnetic flux density ($B$) is strictly analogous to electric current density ($\sigma$), the former being a convenient expression

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for the flux per unit area in a magnetic field, and the latter a convenient expression for the current per unit area in an electric circuit.

From equation (8a), the total flux $\varphi$ through a surface of area $S$, which is normal to the direction of the lines of force, and at each point of which the flux density has the same value $B$, is

$$\varphi = BS \tag{9}$$

When the area $S$ is not perpendicular to the lines of force, the flux through this area is the same as that through the projection of the area $S$ on a plane which is perpendicular to the lines of force, as illustrated in Fig. 48, namely,

$$\varphi = BS' = BS \cos \alpha \tag{9a}$$

where $\alpha$ is the angle between the normal to the area $S$ and the direction of the lines of force.
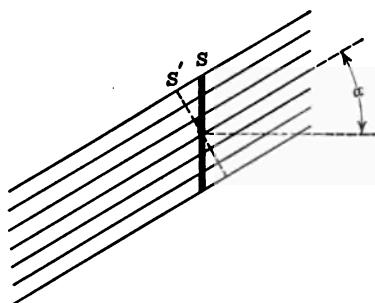


FIG. 48.

When the flux density is different at the various points of the area $S$, and this area is not perpendicular to the lines of force, the total flux through the area is

$$\varphi = \int_s (B \cos \alpha) dS \tag{9b}$$

where the symbol $\int_s$ indicates the summation of the products $(B \cos \alpha)dS$ for all the infinitesimal areas which make up the surface $S$.

The flux density $B$ may, therefore, be considered as a vector quantity which, at any particular point, has the direction of the line of force through that point, and the quantity $B \cos \alpha$ in the

last two expressions may then be looked upon as the component of the flux density $B$ normal to the area considered. A verbal statement of the relation expressed by equation (9$b$) is then that the flux through any area is the integral, over this area, of the normal component of the flux density at this area.

The reader should note carefully, however, that although the flux density at any point is a vector quantity, the *flux through a surface* is an algebraic quantity. That is, the angle between the *direction* of a line of force (*i.e.*, the direction of the flux density) and the normal to a surface at a given point, may have any value, but the *number* of lines of force (*i.e.*, the flux) through this surface from its $A$ face to its $B$ face will be either a positive or negative number, accordingly as these lines pass through this surface from its $A$ face to its $B$ face, or from its $B$ face to its $A$ face (see Article 82). Compare with the relation between force and work: force is a vector quantity, but work, which is the integral along a line of the component of the force along this line, is an *algebraic* quantity.

It is common practice, however, to speak of the direction of a flux, meaning thereby the direction of the lines of force which represent this flux.

In the c.g.s. electromagnetic system of units, the unit of magnetic flux density is 1 maxwell, or line, per square centimeter. This unit is called the "gauss." Flux density is also expressed in lines per square inch and in kilolines per square centimeter, or per square inch. By a kiloline is meant 1000 lines, or 1000 maxwells.

When the magnetic flux density at every point in a given region has the same value, the field is said to be "uniform" within this region. In a uniform magnetic field the lines of force are straight, parallel, and uniformly distributed.

Problem 8.—Referring to Problem 7, what is the average value of the flux density in the air gap over the particular area which is bounded by the coil when placed in the air gap as specified?

*Answer.*—54,700 lines per square inch or 8490 gausses.

Problem 9.—Referring to Fig. 46, the flux density in the air gap directly under the center of each pole is 50,000 lines per square inch, and is perpendicular to the surface of the armature. At the points midway between the two poles the component of the flux density normal to the surface of the armature is zero. At any other point $P$ in the air gap, the component of

the flux density normal to the surface of the armature, is given by the expression[1]

$$B = 50,000 \cos\left(\frac{2\pi x}{25}\right)$$

where 25 is the distance, in inches, measured circumferential around the armature, from the center of a north pole to the center of the next north pole, and $x$ is the distance, in inches, measured circumferentially around the armature, from the center of a north pole to the point $P$. The length of the armature core, measured parallel to the shaft, is 8 inches. The surface of the pole face is rectangular.
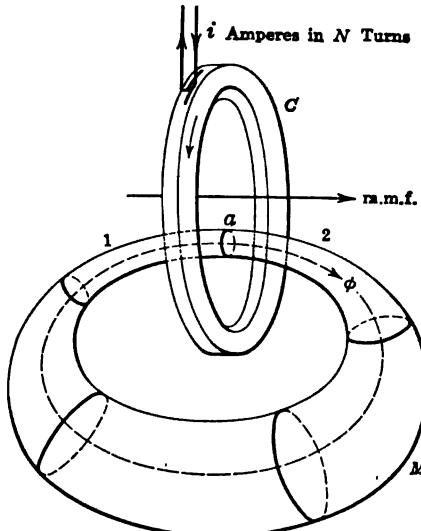


FIG. 49.

(a) What is the total flux entering the armature from each pole? (b) Plot a curve with the flux density $B$ as ordinates and the distance $x$ as abscissas. (c) What is the relation between the area of this curve and this total flux?

*Answer.*—(a) $3.18 \times 10^6$ lines. (c) The flux per pole is equal to this area multiplied by the length of the armature, which in this case is 8 inches.

**88. Magnetomotive Force and Magnetic Reluctance.**—The magnetic fields in most electric machinery and apparatus are due directly to electric currents.[2] The relation between the magnetic

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[1] Such a "sinusoidal" distribution of flux is seldom actually realized in practice, but is often closely approximated, particularly in alternating-current machines.

[2] The magnetic fields of so-called permanent magnets will be considered in detail later; Article 104.

flux produced by an electric current and the intensity of this current is, therefore, one of fundamental importance.

The relation in question is of the same general form as that between an electromotive force and the current which it produces. Consider, for example, the magnetic field due solely to a current $i$ in a single coil of $N$ turns. The space in which the magnetic field exists may be thought of as divided into a number of *closed* rings, such as the ring $M$ in Fig. 49, each ring being of such a shape that its walls are tangent at each point to the line of force through that point. Such rings will not in general be of a simple shape, and the cross-section of any particular ring at various points along its axis will not in general be the same.

Since the walls of such a ring as that described are tangent at each point to the line of force through that point, no line of force can enter or leave it through its walls. Therefore every line of force inside such a volume must be a closed loop intersecting every cross-section of this volume, as illustrated by the dotted loop in Fig. 49. The magnetic flux will therefore have the same value at every cross-section of such a volume. A volume whose walls are tangent at each point to the line of force through that point therefore forms a closed "circuit" for the magnetic flux in it, just as a loop of wire forms a circuit for any electric current which may be established within it. Such a volume may therefore be conveniently designated as the "magnetic circuit" of the flux within it.

The term "magnetic circuit" is also used to designate specifically the iron cores, yokes and air gaps of any electric apparatus or machine, for these parts form the path of the major portion of the magnetic flux. However, as will be shown later, some of the lines of force through the cores and yokes "leak" out through their lateral walls into the surrounding air.

Consider a closed magnetic circuit, such as the ring above described, from whose lateral walls there is no leakage of magnetic flux. Experiment shows that when such a circuit is linked by a *single* coil of $N$ turns, carrying a current $i$, the relation between the flux $\varphi$ in this magnetic circuit and the current $i$ in the given coil may always be represented by an expression of the form

$$\varphi = \frac{Ni}{\mathcal{R}} \qquad (10)$$

where $\Re$ is a factor, called the "magnetic reluctance" of the given magnetic circuit, which factor represents a definite property of this circuit, just as the resistance of a given electric circuit represents a definite property of the electric circuit.

The reluctance of a given magnetic circuit, as thus defined, is found by experiment to depend upon the size, shape and dimensions of this circuit (see Article 89) in exactly the same way that the resistance of the path of an electric current depends upon the size, shape and dimensions of this path. The numerical value of the reluctance of a given magnetic circuit also depends, of course, upon the unit adopted for its measure.

In the c.g.s. electromagnetic system of units, the unit of reluctance is so chosen that when the flux is expressed in maxwells and the current in abamperes, the above relation between the flux, current and reluctance becomes

$$\varphi = \frac{4\pi Ni}{\Re} \tag{10a}$$

The factor $4\pi$ arises from the fact that the c.g.s. electromagnetic system of units is based upon the arbitrary choice of unity as the value of the reluctance of a centimeter-cube of free space (see Article 89). The c.g.s. electromagnetic unit of reluctance is called the "oersted."

The relation expressed by equation (10a) is of identically the same form as that which holds between the current $I$, the electromotive force $E$, and the resistance $r$, of a closed electric circuit, viz., the current in a *closed* electric circuit is equal to the electromotive force in this circuit divided by its resistance, or,

$$I = \frac{E}{r}$$

The quantity $4\pi Ni$ is, therefore, called the "magnetomotive force" (abbreviated m.m.f.) established by the given current $i$ in the magnetic circuit which it links.

Equation (10a) may, therefore, be looked upon as a mathematical statement of the fact that a current of $i$ abamperes in a coil of $N$ turns establishes a magnetomotive force in every magnetic circuit which links each turn of this coil, and that this magnetomotive force produces in any such magnetic circuit a flux numerically equal to this magnetomotive force divided by the reluctance of this circuit, *provided* the magnetic field is due solely to the current in this one coil (see Article 90).

As previously noted (Article 78), the flux lines produced by a single current always link this current in the right-handed screw direction, except in the special case of a core of iron or other magnetic substance which has been previously magnetized, in which case the resultant lines of force may be linked by the current in the opposite, or left-handed screw, direction. Since a magnetomotive force may be looked upon as that which produces, or tends to produce, a flux, the direction of the magnetomotive force established in a magnetic circuit by any current which links it, is taken as that direction through the circuit of this current which bears a right-handed screw relation to the direction of this current. For example, in Fig. 49, the direction of the magnetomotive force is that of the arrow marked m.m.f.

The fact that, in a previously magnetized core of iron (or other magnetic material), the resultant lines of force may link in the left-handed screw direction the current in a winding on this core, may be conveniently taken into account by considering the reluctance of such a core to this resultant flux as a *negative* quantity (see Article 111).

The unit of magnetomotive force in the c.g.s. electromagnetic system is called the "gilbert;" *i.e.*, a gilbert is the magnetomotive force produced by a current of $\frac{1}{4\pi}$ abamperes, or $\frac{10}{4\pi}$ amperes, in a coil of 1 turn. The practical unit of magnetomotive force is the "ampere-turn." By an ampere-turn is meant the magnetomotive force produced by a current of 1 ampere in a coil of 1 turn. A current of $i$ amperes in a coil of $N$ turns produces a magnetomotive force of $Ni$ ampere-turns (see equation (10)). The relation between the ampere-turn and the gilbert is

$$1 \text{ ampere-turn} = 0.4\pi \text{ gilberts}$$
$$= 1.257 \text{ gilberts}$$

When the current $i$ is expressed in amperes, the flux $\varphi$ in maxwells, and the reluctance $\mathfrak{R}$ in oersteds, equation (10*a*) becomes

$$\varphi = \frac{0.4\pi Ni}{\mathfrak{R}} \tag{10*b*}$$

The fundamental relation between magnetomotive force, flux and reluctance will be expressed in this form throughout this chapter.

The reciprocal of the reluctance of a magnetic circuit is called the "magnetic permeance" of this circuit, and is usually designated by the symbol $\mathcal{P}$, namely,

$$\mathcal{P} = \frac{1}{\mathcal{R}} \tag{11}$$

The flux in a magnetic circuit which has a permeance of $\mathcal{P}$ c.g.s. electromagnetic units is then

$$\varphi = \mathcal{P} \times (0.4\pi Ni) \qquad \text{maxwells}$$

when $i$ is expressed in amperes.
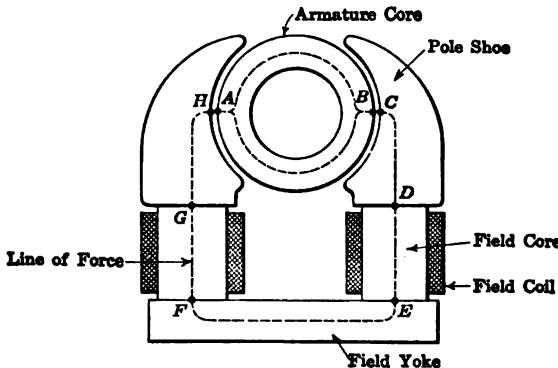


FIG. 50.

**Problem 10.**—In Fig. 50 is shown the magnetic circuit of a two-pole electric generator. Each of the two field coils has 2210 turns and a resistance of 65 ohms. These two coils are connected in series in such a manner that each tends to produce a magnetic flux in the same direction around the magnetic circuit of the machine. 115 volts is impressed on this field winding. The reluctance of the magnetic circuit of the machine is 0.0021 oersted.

(a) What is the magnetomotive force impressed on this circuit, in ampere-turns? (b) In gilberts? (c) What is the value of the total flux established in this circuit, in maxwells? (d) If 20 per cent. of this total flux passes through the air from one pole to the other, without entering the armature core, what will be the flux in the air gap directly under each pole face (*i.e.*, the flux through the armature core)?

*Answer.*—(a) 3910 ampere-turns. (b) 4920 gilberts. (c) 2.34 × 10⁶ maxwells. (d) 1.87 × 10⁶ maxwells.

**89. Magnetic Permeability and Reluctivity.**—The reluctance of a magnetic circuit, like the resistance of an electric circuit, depends not only upon the size and shape of the path which forms this circuit, but also upon the nature of the substances

through which this path passes. This relation may, in fact, be expressed in exactly the same manner as the relation between the resistance of an electric circuit and the length, cross-section and conductivity of the component parts of this circuit.

In the first place, just as certain substances conduct electricity more readily than others, so also are certain substances more "permeable" to a magnetic flux than others. The substance which is most permeable to a magnetic flux is pure iron. The various kinds of commercial iron and steel are also highly permeable to a magnetic flux, and, due to their relative cheapness, are invariably used where a path of low magnetic reluctance (large flux for a small number of ampere-turns) is desired.

Most substances, however, such as air and other insulating materials, copper, aluminum, etc., have practically the same degree of magnetic permeability as free space (*i.e.*, a perfect vacuum). Such substances are, therefore, usually said to be "non-magnetic," as distinguished from those substances which have a permeability of the same order of magnitude as that of iron, which latter are commonly referred to as "magnetic" substances. Aside from iron and steel, the only strongly magnetic substances are nickel and cobalt and certain alloys of manganese, known as Heusler alloys.

Since every substance, and even free space, has a permeability appreciable in magnitude with that of iron, it is impossible to "insulate" the path of a magnetic flux to the degree that the path of an electric current (*e.g.*, a copper wire) can be insulated. Consequently, even when iron or steel of high permeability is used for the path of a magnetic flux, the flux in this main path will, as a rule, be accompanied by a certain amount of "leakage" flux in the air and other non-magnetic substances in the vicinity of this main path. This leakage flux is comparable in magnitude with the current which would leak from an electric circuit made of carbon and immersed in a strong salt solution. The leakage flux about an electric dynamo may be readily detected by the pull produced on a small piece of iron (*e.g.*, a screw-driver) held near it.

Due to the fact that a magnetic flux is practically never confined to a linear path (such as a wire) it is usually impossible to express in a simple formula the exact relation between the reluctance of the path of a given flux and the dimensions and nature

of the substance through which this path passes. However, in the special case of a volume which has the form of a **right cylinder or prism,** and in which the flux has the **same density at every point** and is **parallel to the axis** of this volume, experiment shows that the reluctance may be expressed by the simple formula

$$\mathfrak{R} = \frac{l}{\mu S} \tag{12}$$

where $l$ is the length, and $S$ the cross-section of this volume, and $\mu$ is a factor whose numerical value may be taken as the measure of the magnetic "permeability" of this volume. This relation is identical in form with that between the resistance and conductivity of a volume of like shape to a current which is parallel to its axis and of constant density throughout (see Article 55).

Any magnetic circuit from which there is no leakage of flux, no matter what its shape or size, may be considered as made up of elementary volumes which satisfy the conditions just stated. Its total reluctance may then be calculated by combining the reluctances of these elementary volumes in the same manner as resistances in series and in parallel are combined.

As noted in the last article, the c.g.s. electromagnetic system of units is based on the arbitrary choice of unity as the value of the reluctance of a centimeter-cube of free space. From equation (12) this is equivalent to the arbitrary choice of unity as the value of the magnetic permeability of free space. Magnetic permeability is almost invariably expressed in c.g.s. electromagnetic units, and will be so expressed throughout this book.

Experiment shows that certain substances have a magnetic permeability slightly less than that of free space. Such substances are called "diamagnetic" substances. Bismuth has the smallest permeability of any substances, namely, a permeability of 0.99983. Those substances which have a permeability but slightly greater than unity are called "paramagnetic" substances. Substances such as iron, steel, nickel, cobalt and the Heusler alloys, which have a permeability many times greater than air or free space, are called "ferromagnetic" substances. For most practical purposes, the permeability of all substances other than these paramagnetic substances may be taken as unity.

A fact of fundamental importance in regard to the permeability of any strongly magnetic substance, such as iron or steel, is

that its permeability depends not only (1) upon its chemical composition, but also (2) upon the flux density established in it, (3) its previous magnetic history, and (4) the heat treatment to which it has been subjected. In particular, a permanent magnet must be looked upon as having a *negative* permeability to its own flux. In fact, a core of any magnetic material may, under certain conditions, have a negative permeability. These various factors are considered in detail in Chapter IX.

The maximum normal permeability of pure iron is about 13,000. However, for such qualities of sheet iron and steel as are ordinarily used in electric machinery and apparatus, at the flux densities commonly employed, the permeability is usually of the order of 3000 c.g.s. electromagnetic units. Cast steel and cast iron have a much lower normal permeability (see Article 112).

The reciprocal of magnetic permeability is often a more convenient quantity to employ than the permeability itself. This reciprocal, namely, the quantity

$$\rho = \frac{1}{\mu} \tag{13}$$

is called the "reluctivity" of the material in question. Reluctivity is analogous to electric resistivity, which later is the reciprocal of the electric conductivity (see Article 58). The student should not be confused by the use of the same symbol, $\rho$, for reluctivity and for resistivity. A distinctive symbol for reluctivity would be desirable, but it has become common practice to use the same symbol for each. Whenever there is a possibility of confusion, the letter $\rho$ with the subscript "$m$" may be used to designate reluctivity, viz., $\rho_m$.

Values of the "normal" reluctivity of iron, steel, cobalt and nickel are given by the curves in Fig. 77, which are more fully described in Article 112.

**Problem 11.**—A rectangular bar of iron 10 inches long, 1.2 inch wide and 0.5 inch thick has a reluctance of 0.0045 oersteds, when the flux through it is parallel to its axis, uniformly distributed, and equal in value to 30,000 maxwells.

(a) What is the flux density in the bar, in gausses? (b) What is the permeability of the bar to a flux of this density? (c) What is the corresponding value of the reluctivity?

*Answer.*—(a) 7760 gausses. (b) 1457 c.g.s. electromagnetic units. (c) 0.000686 c.g.s. electromagnetic units.

**90. Magnetizing Force.**—As noted in the previous article, equation (12) is the basis for the calculation of the reluctance of all magnetic circuits, irrespective of their shape and of the distribution of flux within them, just as the corresponding expression for the resistance of a conducting rod or bar is the basis for the calculation of the resistance of a conductor of any shape and dimensions (see Article 64). However, since a magnetic flux is practically never confined to a straight cylindrical path, such as a rod or wire, the formula

$$\mathfrak{R} = \frac{l}{\mu S}$$

when applied directly to the calculation of the reluctance of a *finite* volume in a magnetic field, will, as a rule, give only an approximately correct value of the reluctance of this volume. Nevertheless, such rough calculations are often sufficiently accurate for engineering purposes.

For example, the reluctance of the field yoke in the dynamo shown in Fig. 50, to the flux which passes through it from one field core to the other, may be calculated from this formula, with a sufficient degree of accuracy, by taking for $l$ the length of the central line of force in this yoke (*i.e.*, the length $EF$ measured along the dotted line in Fig. 50), for $S$ the actual cross-section of the yoke, and for $\mu$ the permeability of the yoke corresponding to the value of the flux density which it is desired to establish in it.

From an engineering point of view, the reluctance of a magnetic circuit is chiefly of importance in that it determines the number of ampere-turns which must link this circuit in order to establish a given flux in it. A more direct calculation of the ampere-turns required may, however, be made without actually calculating the reluctance, by proceeding in a somewhat different manner, which will now be described. The procedure is identical with that employed in calculating the drop of electric potential in a non-linear conductor (see Article 64).

From the definition of magnetomotive force and reluctance (Article 88), the magnetomotive force which acts on any closed magnetic circuit from which there is no leakage of flux (see Fig. 49) is equal to the flux $\varphi$ through this circuit multiplied by its total reluctance $\mathfrak{R}$, viz.,

$$0.4\pi Ni = \mathfrak{R}\varphi \qquad (14)$$

where the magnetic quantities $\mathcal{R}$ and $\varphi$ are expressed in c.g.s. electromagnetic units and the current $i$ in amperes.

This total reluctance $\mathcal{R}$ may be considered, as noted in Article 89, as the sum of the reluctances, say $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, etc., of the various sections which make up the complete circuit of the flux $\varphi$. Whence equation (14) may be written

$$0.4\pi Ni = \mathcal{R}_1\varphi = \mathcal{R}_2\varphi + \mathcal{R}_3\varphi + \text{etc.} \qquad (14a)$$

where $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, etc., are the reluctances of the various sections which, in series, make up the complete circuit of the flux $\varphi$.

Consider a closed magnetic circuit of infinitesimally small cross-sections, and let $d\varphi$ be the flux in this circuit. Let $dS$ be the cross-section of this circuit at any point on its axis. This axis will be a line of force, such as the dotted loop in Fig. 49. Let $dB$ be the flux density at the point at which $dS$ is taken, and let $dl$ be an elementary length measured along the axis from this point. Then the volume $dl \times dS$ forms a right cylinder (or prism) in which the flux density may be considered as constant. The axis of this cylinder is parallel to the direction of the flux through it. Hence, from equation (12), the reluctance of this elementary volume is

$$d\mathcal{R} = \frac{dl}{\mu dS}$$

where $\mu$ is the permeability of the medium at the point considered.

The product of this reluctance by the flux $d\varphi$ through it is

$$d\mathcal{R} \times d\varphi = \frac{dl}{\mu}\frac{d\varphi}{dS}$$

But $\frac{d\varphi}{dS}$ is equal to the flux density $B$ at the point considered (see equation (8)); whence

$$d\mathcal{R} \times d\varphi = \frac{B}{\mu}dl \qquad (15)$$

From equation (14a), however, the summation of $d\mathcal{R} \times d\varphi$ for all the elementary volumes which constitute the given magnetic circuit is equal to $0.4\pi Ni$, which in turn, from equation (15), must be equal to the summation of $\frac{B}{\mu}dl$ for all the elementary lengths which make up the line of force which coincides with this magnetic circuit. Whence, equation (14) is equivalent to the relation

$$0.4\pi Ni = \int_L \frac{B}{\mu} dl \tag{16}$$

where the integral sign $\int_L$ is used to indicate that the quantity $\frac{B}{\mu}$ is integrated completely around the loop formed a line of force.

The quantity $\frac{B}{\mu}$, *i.e.*, **the magnetic flux density at any point divided by the magnetic permeability at this point, when both are expressed in c.g.s. electromagnetic units, is called the "magnetizing force"** at this point, and is usually represented by the symbol $H$, viz.,

$$H = \frac{B}{\mu} \qquad \text{c.g.s. electromagnetic units} \tag{17}$$

Or, in terms of the reluctivity $\rho$,

$$H = \rho B \qquad \text{c.g.s. electromagnetic units} \tag{17a}$$

Compare with equation (16) of Article 64.

In terms of the magnetizing force $H$, as thus defined, equation (16) of this article may be then written

$$0.4\pi Ni = \int_L H dl \tag{18}$$

The student should note carefully that the terms "magnetomotive force" and "magnetizing force" are used for two entirely different quantities; namely, the *magnetomotive* force established in a closed loop in equal to $0.4\pi$ times the ampere-turns which link this loop, whereas the *magnetizing* force at any point on this loop is the flux density which is established at this point divided by the permeability of the medium at this point. Equation (18) states that these two quantities bear such a relation to each other, that **the integral of the magnetizing force around the closed loop formed by any line of force is equal to the magnetomotive force established in this loop by the ampere-turns which link it.**

Magnetizing force, therefore, has the same "dimensions" as magnetomotive force divided by length. Consequently, since in the c.g.s. electromagnetic system of units, the unit of magnetomotive force is the gilbert, the c.g.s. electromagnetic unit of magnetizing force may be called the "gilbert per centimeter."

Since, by definition, $H = \dfrac{B}{\mu}$, equation (15) may be written

$$d\mathcal{R} \times d\varphi = H\,dl \tag{19}$$

That is, the product of the magnetizing force at any point by an elementary length of the line of force through this point, is equal to the reluctance of the elementary cylinder, whose axis is this elementary length, multiplied by the flux through this cylinder. From analogy with the product of a resistance by an electric current, the product of the reluctance of any given
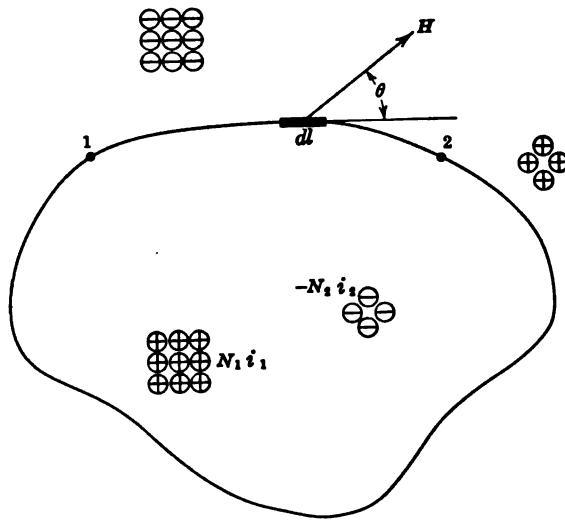


Fig. 51.

volume by the flux through it may be called the "reluctance drop" in this volume. The product $H\,dl$, where $dl$ is an elementary length in a line of force, is then the *reluctance drop* along this elementary length.

Equation (18) is then merely a convenient mathematical expression for the relation that the total reluctance drop around a closed magnetic circuit is equal to the magnetomotive force established in this circuit. This, in fact, is the relation (see equation (14$a$)) from which equation (18) is deduced. Compare also with the relation between the total resistance drop around a *closed* electric circuit and the electromotive force in this circuit.

Experiment shows that the magnetizing force $H$ at any point, like the flux density $B$ at a point, may be considered as a vector quantity, whose direction is along the line of force. When the

permeability at the given point is positive, which is usually the case, the positive sense of the magnetizing force $H$ is that of the line of force at this point. When the permeability at the given point is negative (see Article 111), the positive sense of the magnetizing force $H$ must then be taken *opposite* to that of the line of force. The positive sense of the flux density $B$, however, is always in the direction of the line of force.

Equation (18) may then be put into a more general form, which experiment shows to be applicable to *any path whatever*, irrespective of how the magnetic field is produced, whether by electric currents, magnets, or both, viz.,

$$\int_L (H \cos \theta) dl = 0.4\pi \Sigma Ni \qquad (20)$$

where $dl$ represents an elementary length, in centimeters, measured along this path (see Fig. 51); $H$ is the resultant magnetizing force at $dl$, in gilberts per centimeter; $\theta$ is the angle between the direction of $H$ and the direction of $dl$; and the symbol $\int_L$ signifies the summation of the products $(H \cos \theta) dl$ for all the elementary lengths into which this path is divided, and $\Sigma Ni$ signifies the algebraic sum of all the ampere-turns which link this loop. In this summation those currents which link the given loop in the right-handed screw direction, with respect to the direction in which $dl$ is measured, are to be considered as positive, and those currents which link this loop in the opposite direction are to be considered as negative.[1]

In general, the integral along any given path from any point $A$ to any other point $B$, of the component, in the direction of this path, of any vector quantity $X$, is called the "line integral" of this quantity along this path. That is, the line integral of $X$ along any given path from $A$ to $B$, is

$$\int_A^B (X \cos \theta) dl$$

where $\theta$ is the angle between the direction of $X$ and the direction of $dl$. A familiar case of a line integral is the expression for the work $W$ done by a force $F$ in moving a particle of matter along a given path, viz.,

$$W = \int_A^B (F \cos \theta) dl$$

---

[1] This relation is perfectly general only when the "displacement currents" as well as the conduction currents which link the loop are included in this summation (see Article 137).

The general relation expressed by equation (20) may then be stated in words, viz., *the line integral of the magnetizing force around any closed path in a magnetic field is always equal to the algebraic sum of the magnetomotive forces which act on this path.*

When the closed path around which the line integral is taken links no electric current, equation (20) becomes

$$\int_L (H \cos \theta) dl = 0 \qquad (20a)$$

That is, the line integral of the magnetizing force around a closed path which links no electric current is zero. In particular, the line integral of the magnetizing force due *solely* to a permanent magnet, around a closed line of force which passes through this magnet, is always *zero*

Many physicists use the term "magnetic field intensity" to designate the quantity above defined as the "magnetizing force," and express its magnitude as so many "dynes per unit north pole." The name "magnetizing force" for the quantity $H$, and the name "gilbert per centimeter" for its unit (in the c.g.s. electromagnetic system), are much more suggestive of the real significance of this quantity.

Since the permeability of air and other non-magnetic substances is equal to unity (practically), when expressed in c.g.s. electromagnetic units, the magnetizing force in such substances is always numerically equal to the flux density, provided both quantities are expressed in c.g.s. electromagnetic units. On this account, magnetizing force, in the c.g.s. electromagnetic system, is frequently stated as so many gausses, or maxwells per square centimeter, or lines per square centimeter. However, only on the assumption that the permeability has no dimensions (*i.e.*, is merely a number, like the factor $\pi$, for example), is it logical to express flux density ($B$) and magnetizing force ($H$) in the same units. Such an assumption may be correct, but it will avoid confusion to reserve the terms "gauss," "maxwells per square centimeter" and "lines per square centimeter" to designate the flux density, and to designate the c.g.s. electromagnetic unit of magnetizing force by the specific term "gilbert per centimeter."

As noted in Article 88, the practical unit of magnetomotive force is the ampere-turn. The corresponding practical unit of magnetizing force is the ampere-turn per centimeter, or per inch,

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accordingly as the centimeter or inch is used as the unit of length. When $H$ is so expressed, the factor $0.4\pi$ in equations (18) and (20) must be omitted. The general relation (equation (20)) between magnetizing force and magnetomotive force, when the closed loop $L$ is a line of force, is, therefore, in practical units,

$$\int_{L} H dl = \Sigma Ni \qquad (21)$$

where $dl$ is an elementary length, in centimeters (or inches), measured along this line of force, $H$ is the magnetizing force at $dl$ in ampere-turns per centimeter (or per inch) and $\Sigma Ni$ is the algebraic sum of the ampere-turns linked, in the right-handed screw direction, by this line of force. This equation is the basis



This Winding continuous all the way around Ring, with Turns Uniformly spaced
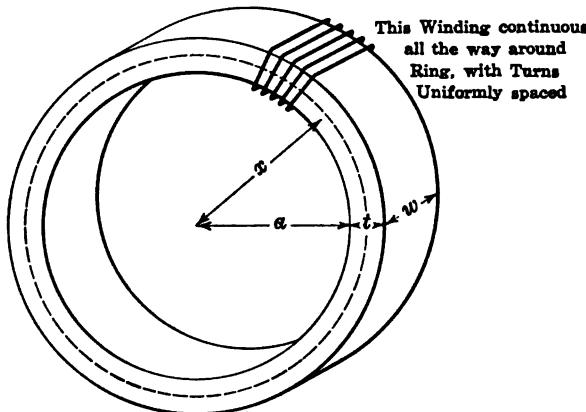
FIG. 52.

for the practical calculation of the magnetic circuits of electric machines and apparatus (see the next article).

Ampere-turns per centimeter or per inch may be readily converted into gilberts per centimeter from the relations that

1 ampere-turn per inch       = 0.4950 gilberts per centimeter
1 ampere-turn per centimeter   = 1.257   gilberts per centimeter

It should be carefully noted, that when the flux density $B$ is expressed in gausses, $\mu$ in c.g.s. electromagnetic units, and $H$ in ampere-turns per inch, the relation between $B$, $\mu$ and $H$ (equation (17)) becomes

$$H = 2.021\frac{B}{\mu} \qquad \text{ampere-turns per inch} \qquad (22)$$

Similarly, when $B$ is expressed in lines per square inch, $\mu$ in c.g.s. electromagnetic units, and $H$ in ampere-turns per inch, the relation between $B$, $\mu$ and $H$ becomes

$$H = 0.3131\frac{B}{\mu} \qquad \text{ampere-turns per inch} \qquad (22a)$$

**Problem 12.**—In Fig. 52 is shown an iron ring on which is uniformly wound, as indicated in the figure, a coil of 600 turns of insulated wire. The radius of the hole in the ring, marked $a$ in the figure, is 5 inches, the radial thickness $t$ is ¾ inch, and the axial width $w$ is 2 inches. A current of 0.7 ampere is established in the coil, and produces in the ring a flux of 100,000 maxwells. Due to the fact that the ring is *completely* covered with a *uniform* winding, the lines of force produced by the current in this winding must, from symmetry, be confined to space inside the winding and must be circles concentric with the hole in the ring (see Article 108).

(a) What is the magnetomotive force, in ampere-turns and in gilberts, acting on each line of force in the ring? (b) What is the magnetizing force, in ampere-turns per inch and in gilberts per centimeter, at any point on the line of force which has a radius of $x$ inches. (c) What is the maximum and what is the minimum value of this magnetizing force, and at what points do these values occur? (d) Determine by integration with respect to the distance $x$, *i.e.*, by the formula

$$\text{Average value of } H = \frac{1}{t}\int_a^{a+t} H\,dx$$

the *average* value of the magnetizing force over this cross-section? (e) What is the value of the magnetizing force at the mean circumference of the ring (radius $5 + ½ \times 0.75$)? (f) Under what conditions will the *average* value of the magnetizing force be practically equal to the magnetizing force at this mean circumference? (g) What is the average flux density, in gausses, over any cross-section of the ring? (h) What is the average permeability of the ring, taken as the quotient of this average flux density by the average magnetizing force? (i) What is the total reluctance of the ring, in oersteds, as calculated from the fundamental definition of reluctance as the magnetomotive force divided by the flux? (j) Calculate the reluctance from the formula $R = \dfrac{l}{\mu S}$, using for $l$ the mean circumference of the ring. (k) Under what conditions will the formula $R = \dfrac{l}{\mu S}$ give a practically correct value for the reluctance of the ring? (l) Were the ring made of wood instead of iron, how would the magnetizing force in it differ from that in the iron ring, for the same current (namely 0.7 ampere) in the winding? (m) What would be the average flux density in the wooden ring? (n) What would be the total flux through the wooden ring? (o) To produce in the wooden ring the same total flux (100,000 lines) as is produced in the iron ring, how many amperes would be required in the coil? (p) If the current required to magnetize the iron core develops 1 watt of heat in the winding (due to its resistance), how much heat would be developed by the current required to establish an equal flux in the wooden ring?

*Answer.*—(a) 420 ampere-turns or 528 gilberts. (b) $\dfrac{66.9}{x}$ ampere-turns per inch or $\dfrac{33.1}{x}$ gilberts per centimeter. (c) Maximum value 6.62 gilberts per centimeter at the inside surface of the ring ($x = 5$ inches) and a minimum value of 5.76 gilberts per centimeter at the outside surface ($x = 5.75$ inches). (d) 6.16 gilberts per centimeter. (e) 6.15 gilberts per centimeter. (f) The radial thickness $t$ of the ring must be small in comparison with the radius $a$ of the hole in it. (g) 10,320 gausses. (h) 1678 c.g.s. electromagnetic units. (i) 0.00528 oersteds. (j) 0.00529 oersteds. (k) The radial thickness $t$ of the ring must be small in comparison with the radius $a$ of the hole in the ring. (l) For the same number of ampere-turns, the magnetizing force would be the same in the wooden ring as in the iron ring. (m) 6.16 gausses, equal to the average value of the magnetizing force. (n) 59.6 lines. (o) 117.5 amperes. (p) 2,820,000 watts or 2820 *kilowatts*. (This brings out strikingly the advantage of using iron instead of wood, or other non-magnetic material, for the magnetic circuits of electric apparatus.)

**91. Practical Calculation of the Ampere-turns Required to Produce a Given Flux.**—The following example will illustrate the application of the ideas above developed, to the calculation of the ampere-turns required to establish a given flux in the magnetic circuit of an electric machine. Consider a two-pole dynamo, such as shown in Fig. 50. Let

$l_g$ = length of each air gap between the pole face and the armature, viz., $BC = HA$ in Fig. 50.

$l_s$ = mean length of the path of the flux in each pole shoe, viz., $CD = HG$ in Fig. 50, measured along the dotted line.

$l_c$ = length of the path of the flux in each pole core, viz., $DE = GF$ in Fig. 50.

$l_y$ = mean length of the path of the flux in the yoke, viz., $EF$ in Fig. 50, measured along the dotted line.

$l_a$ = mean length of each branch of the path of the flux in the armature core, viz., $AB$ in Fig. 50, measured along either the upper or lower dotted line.

$S_g$ = cross-section of the path of the flux through the air gap, which, to a first approximation, may be taken as the area of the pole face.

$S_s$ = mean cross-section of the path of the flux in the pole shoe; to be estimated from the dimensions of the latter and the general shape of the lines of force through it.

$S_c$ = cross-section of the field core.

$S_y$ = mean cross-section of the path of the flux in the yoke; to be estimated in the same manner as $S_s$.

$S_a$ = cross-section of ring which forms the armature core. The total area of the path of the flux through the armature will then be $2S_a$.

$\varphi$ = total flux through the air gap and armature core.

$k\varphi$ = total flux through the pole shoes, field cores and yoke, where $k$ is a factor (usually between 1.1 and 1.4) which takes into account the leakage flux which passes through the air from one pole shoe to the other, without going through the armature core and air gap. This factor is called the "leakage factor," and may be defined as the ratio of the flux in the field cores to the flux in the armature core.

First calculate the flux densities in the various parts of the magnetic circuit. In the air gap and armature these flux densities are respectively

$$B_g = \frac{\varphi}{S_g} \quad \text{and} \quad B_a = \frac{\varphi}{2S_a}$$

In the pole shoes, field cores and yoke the flux densities are respectively

$$B_s = \frac{k\varphi}{S_s}, \quad B_c = \frac{k\varphi}{S_c} \quad \text{and} \quad B_y = \frac{k\varphi}{S_y}$$

From curves, obtained by test (see Article 112), showing the relation between the flux density $B$ and magnetizing force $H$ for samples of the iron or steel used for the various parts of the magnetic circuit, find the values of the magnetizing force corresponding to the flux densities $B_a$, $B_s$, $B_c$ and $B_y$. Let these values be $H_a$, $H_s$, $H_c$, and $H_y$ respectively. For engineering calculations these curves are usually plotted with flux densities in lines per square inch, and magnetizing forces in ampere-turns per inch. The dimensions of the magnetic circuit are then to be expressed in inches. The magnetizing force in the air gap will then be (see equation (22a))

$$H_g = 0.3131B_g$$

From equation (21) the ampere-turns required will then be

$$NI = 2H_g l_g + 2H_s l_s + 2H_c l_c + H_y l_y + H_a l_a \qquad (23)$$

or the ampere-turns per pole will be

$$\tfrac{1}{2}NI = H_g l_g + H_s l_s + H_c l_c + \tfrac{1}{2}H_y l_y + \tfrac{1}{2}H_a l_a \quad (23a)$$

These last two equations also hold when the dimensions are ex-

pressed in centimeters, provided the magnetizing forces are expressed in ampere-turns per centimeter. When centimeters, however, are used, and the flux densities are expressed in gausses, or lines per square centimeter, the magnetizing force in the air gap is

$$H_g = 0.7958B_g$$

It should be noted that the product $Hl$ in equations (23) or (23a), for any particular section of the magnetic circuit, is equal to the reluctance drop in this section, expressed in ampere-turns. The corresponding reluctance drop $\Re\varphi$ in gilberts is $0.4\pi Hl$. Hence the reluctance, in oersteds, of any particular section of the circuit may be calculated from the relation

$$\Re = \frac{0.4\pi Hl}{\varphi} \tag{24}$$

where $\varphi$ is the flux through this particular section, in maxwells.

**Problem 13.**—In the dynamo shown in Fig. 50 the mean lengths of the path of the flux in the various parts of the machine are:

| | |
|---|---|
| $BC = \frac{3}{32}$ inch | $FE = 20$ inches |
| $CD = 8$ inches | $AB = 15$ inches |
| $DE = 6$ inches | |

The cross-sections of these paths are respectively:

| | |
|---|---|
| $S_g = 35$ square inches | $S_y = 27$ square inches |
| $S_s = 25$ square inches | $S_a = 12$ square inches |
| $S_e = 22$ square inches | |

The armature core is made of sheet steel laminations and the pole shoes, field cores and yoke are soft steel castings. The relation between the flux density and the magnetizing force for this sheet steel and cast steel may be taken from the curves in Fig. 76. The leakage factor of this machine is 1.3.

(a) How many ampere-turns per pole are required to produce a flux of $1.5 \times 10^6$ maxwells in the air gap of this machine? (b) What is the total reluctance drop in the two air gaps in series? (c) In the remainder (i.e., the iron position) of the magnetic circuit? (d) What proportion of the total ampere-turns are required to overcome the air-gap reluctance? (e) What is the total reluctance of the two pole shoes, two field cores and the yoke, in series? (f) What is the reluctance of the armature? (g) What is the total reluctance of the path of the leakage flux? (Note that this path is in parallel with the path through the two air gaps and armature.) (h) What is the total reluctance of the path of the total flux, including the leakage flux?

*Answer.*—(a) 2120 ampere-turns per pole. (b) 2520 ampere-turns or 3170 gilberts. (c) 1720 ampere-turns or 2160 gilberts. (d) 59 per cent.

(e)  0.00107  oersted.  (f)  0.00005  oersted.  (g)  0.0072  oersted.  (h) 0.00273 oersted.

**92. Difference of Magnetic Potential.**—As pointed out in the preceding article, the total reluctance drop (*i.e.*, the line integral of the magnetizing force $H$) around any *closed loop*, or complete magnetic circuit, is always equal to $0.4\pi$ times the algebraic sum of the ampere-turns which are linked by this loop. The reluctance drop in any particular section of a magnetic circuit, however, is not in general equal to the magnetomotive force of the coil which surrounds this particular section, but may be either less than, or greater than, this magnetomotive force.  For example, the reluctance drop through the field cores of a dynamo is not equal to the $0.4\pi$ times the ampere-turns on these field poles, but is very much less than this.  The major portion of the reluctance drop is in the air gap (see Problem 13), which is actually outside the space surrounded by the field winding.

The difference between the magnetomotive force due to the ampere-turns which are threaded by any particular section of a magnetic circuit, and the reluctance drop through this particular section, is called the "difference of magnetic potential" between the two ends of this section of the circuit.  Or, more precisely, by the difference of magnetic potential between any two points 1 and 2 (see Fig. 51) is meant the algebraic difference between the magnetomotive force due to the ampere-turns threaded by any path from 1 to 2, and the line integral, from 1 to 2, of the magnetizing force along this particular path.

The word "thread" is here used in a specific sense.  In the particular case of a single closed loop which lies in a plane, a line (straight or curved) is said to thread this loop when it passes through the plane area enclosed by it.  A line which lies wholly on either side of the plane of a loop does not thread this loop, nor does a line which passes through this plane at any point *outside* the area enclosed by it.  In the case of a single closed loop which does not lie in a plane (*i.e.*, when the loop is a skew curve), a more general definition is necessary.  This definition is that a line is said to thread such a loop when it passes through a point at which the solid angle subtended by the loop is equal to $2\pi$ (see Article 99).  When a coil has more than one turn and the successive turns are close together, each turn may be considered, to a close degree of approximation, equivalent to a closed loop

occupying the mean position of this turn. A line which threads any one of these equivalent loops is then said to thread the corresponding turn.[1]

As noted in Article 79, any surface which is perpendicular at each point to the magnetic line of force through that point is called a "magnetic equipotential surface." Consider in a magnetic circuit any volume whose lateral walls are tangent at each point to the line of force through that point, and whose end-surfaces are magnetic equipotential surfaces (*e.g.*, the section of the ring $M$ in Fig. 49 whose two end-surfaces are on opposite sides of the coil). Let $\varphi$ be the flux through this volume and let $\mathfrak{R}$ be its reluctance. Let $Ni$ be the number of ampere-turns threaded, in the *right-handed* screw direction, by each line of force in this volume. Then by definition, the difference of magnetic potential between the two ends of this volume is

$$U = 0.4\pi Ni - \mathfrak{R}\varphi \qquad (25)$$

When this quantity is positive, the end of the volume at which the flux enters is said to be at the *lower* magnetic potential, *i.e.*, equation (25) gives the *rise* of magnetic potential in the direction of the flux.

Note that the relation expressed by this equation is identical in form with that between the terminal voltage $v$, the electromotive force $e$ and the internal resistance drop $ri$ for any portion of an electric circuit which is a *source* of electric energy (see Article 40).

When the current threaded by the flux $\varphi$ is in the *left-handed* screw direction with respect to this flux, *i.e.*, when the current produces a counter, or back, magnetomotive force, the current $i$ in equation (25) is a negative quantity, and, therefore, the rise of magnetic potential in the direction of the flux is *negative*. This means that in the direction of the flux $\varphi$ there is an actual *drop* of magnetic potential equal to

$$U = 0.4\pi Ni + \mathfrak{R}\varphi \qquad (25a)$$

---

[1] Strictly, the entire electric circuit of which the coil is a part must be considered. Imagine this circuit to be projected radially on the surface of a sphere of unit radius drawn about any given point $P$. This projection will form as many closed loops on the surface of the sphere as there are turns in the circuit. When any one of these loops has an area of $2\pi$, any line which passes through the point $P$ and through this area is said to thread the particular turn corresponding to this loop.

where $Ni$ is here used to designate the ampere-turns which are threaded by the flux $\varphi$ in the *left-handed* screw direction.

Note that the relation expressed by this equation is identical in form with that between the impressed voltage $v$, the back electromotive force $e$, and the internal resistance drop $ri$ in any portion of an electric circuit which is a *receiver* of electric energy.

For any section of a magnetic circuit which does not thread an electric circuit, equation (25a) becomes

$$U = \Re\varphi \qquad (25b)$$

That is, when the flux from any magnetic equipotential surface $A$ to any other magnetic equipotential surface $B$ threads no ampere-turns, the drop of magnetic potential in the direction of the flux is equal to the product of this flux by the reluctance of its path.   Hence the name "reluctance drop" for the product $\Re\varphi$.

Note that the relation expressed by equation (25b) is identical in form with Ohm's Law for the electric circuit.   This relation is, therefore, sometimes referred to as "Ohm's Law for the magnetic circuit."

In applying the formulas given in this article, it is often convenient to express the reluctance drop $\Re\varphi$ in terms of the line integral of the magnetizing force $H$ between the two equipotential surfaces which form the ends of the section of the magnetic circuit under consideration.   From the definition of magnetizing force, the reluctance drop $\Re\varphi$ along a line of force from any magnetic equipotential surface 1 to any other magnetic equipotential surface 2 is always

$$\Re\varphi = \int_1^2 H dl \qquad . \qquad (26)$$

where $dl$ is an elementary length in this line of force, $H$ is the value of the magnetizing force at $dl$, and the integral sign indicates the summation of the products $Hdl$ for all the elementary lengths into which this particular section of the line of force is divided.

Note that equation (26) is identical in form with the general expression (Article 64) for the resistance drop in a given portion of an electric circuit, viz.,

$$ri = \int_1^2 F dl$$

where $F$ is the intensity of the electric force. Also note that $H = \dfrac{B}{\mu}$, where $B$ is the flux density and $\mu$ is the magnetic permeability; and $F = \dfrac{\sigma}{\gamma}$ where $\sigma$ is the current density and $\gamma$ the electric conductivity.

When the given points 1 and 2 are not on the same line of force, the reluctance drop from 1 to 2 along any specified path is

$$\Re\varphi = \int_1^2 (H \cos\theta)dl \qquad (26a)$$

where $dl$ is any elementary length in this particular path and $\theta$ is the angle between the direction of $dl$ and the direction of the magnetizing force at $dl$ (see Fig. 51).

From the definition of the difference of magnetic potential above given, and the fundamental relation expressed by equation (20), it may readily be shown that between any two points in a magnetic field there can exist at any instant but a single value of the magnetic potential difference, *i.e.*, magnetic potential is a single valued quantity, just as electric potential is a single valued quantity.

The unit of magnetic potential difference is the same as that of magnetomotive force, viz., the gilbert, in the c.g.s. electromagnetic system of units; and the ampere-turn, in the practical system of units.

Problem 14.—The axial length (or thickness) of a coil of $N$ turns is negligible in comparison with its diameter. The current in the coil is $i$ amperes.

(a) What is the difference of magnetic potential between the two faces of this coil? (b) What is the reluctance drop from one face to the other along any path which *does not thread* this coil? (c) What is the reluctance drop *through the coil* from one face to the other?

Answer.—(a) $0.4\pi Ni$ gilberts. (b) $0.4\pi Ni$ gilberts. (c) Negligible, since the length of the path of the flux through the coil is negligible in comparison with the cross-section of this path.

93. Mutual and Leakage Fluxes.—The significance of the relations developed in the preceding article is well illustrated in the case of two coils which link a common iron core, as shown in Fig. 53. The rectangular ring in this figure represents a closed iron core, on the vertical legs of which are wound two coils, as shown, which are electrically independent, *i.e.*, insulated from each other and from the core. As will be

shown later, an alternating-current transformer is essentially an arrangement of this kind, the only difference being that the two coils $C_1$ and $C_2$ are usually divided into a number of parts, and half of each coil is placed on each leg of the core.

Consider the particular case when the currents in the two coils at any instant are in opposite directions. This is indicated in the figure by plus and minus signs in the cross-sections of the conductors which form the windings, a current in the direction away from the eye of the reader being indicated by a plus sign,
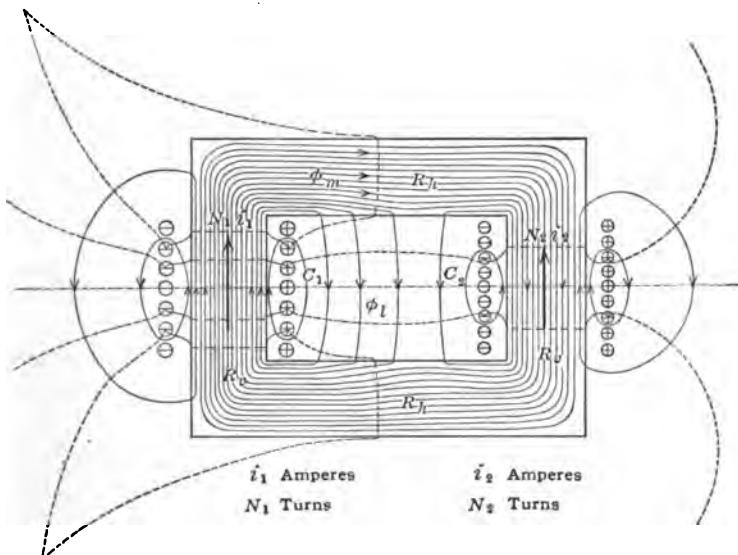


FIG. 53.—Iron core with two coils.

and a current in the direction toward the eye of the reader being indicated by a minus sign. This convention for showing the direction of a current in the cross-section of a conductor is used throughout this book.

Let the ampere-turns $N_1 i_1$ of the "primary" coil $C_1$ be greater than the ampere-turns $N_2 i_2$ of the "secondary" coil $C_2$. The flux lines established by the currents in these two coils will then be as shown in the figure. In particular, those flux lines $\varphi_m$ which link both coils will be in such a direction around the core, as shown by the arrows on them, that they link the current in the primary coil in the right-handed screw direction and the current in the secondary coil in the left-handed screw direction.

Hence the ampere-turns of the secondary coil produce a counter magnetomotive force with respect to this "mutual" flux $\varphi_m$.

Let $\mathfrak{R}_v$ be the reluctance of the path of this mutual flux in each vertical leg, and $\mathfrak{R}_h$ the reluctance of the path of this mutual flux in each horizontal leg or yoke. Then, through the left-hand leg there is a *rise* of magnetic potential of $(0.4\pi N_1 i_1 - \mathfrak{R}_v \varphi_m)$ gilberts, the bottom of this leg being at the lower magnetic potential. Similarly, through the right-hand leg there is a drop of magnetic potential equal to $(0.4\pi N_2 i_2 + \mathfrak{R}_v \varphi_m)$ gilberts, the bottom of this leg likewise being at the lower magnetic potential. In the upper yoke there is a drop of magnetic potential of $\mathfrak{R}_h \varphi_m$ gilberts to the right, and in the lower yoke there is a drop of magnetic potential of $\mathfrak{R}_h \varphi_m$ gilberts to the left.

The resultant drop of magnetic potential completely around the closed core must be zero, for the magnetic potential at a point can have but a single value. Hence the resultant flux will have the value

$$\varphi_m = \frac{0.4\pi(N_1 i_1 - N_2 i_2)}{2(\mathfrak{R}_v + \mathfrak{R}_h)} \tag{27}$$

Compare with the current in a circuit of total resistance $r$ when there exists in the circuit two electromotive forces $E_1$ and $E_2$ which oppose, or "buck," each other.

The average difference of magnetic potential between the upper and lower horizontal yokes will be one-half the sum of $(0.4\pi N_1 i_1 + \mathfrak{R}_v \varphi_m)$ and $(0.4\pi N_2 i_2 - \mathfrak{R}_v \varphi_m)$, or

$$\frac{0.4\pi(N_1 i_1 + N_2 i_2)}{2}$$

Let $\mathfrak{R}_l$ be the total reluctance of the air between and surrounding these two yokes. Then through the air there will be a leakage flux of

$$\varphi_l = \frac{0.4\pi(N_1 i_1 + N_2 i_2)}{2\mathfrak{R}_l} \tag{27a}$$

In other words, with such an arrangement of coils as shown in the figure, there will always be, in addition to the main flux $\varphi_m$, an appreciable leakage flux through the air. It is to reduce this leakage flux that, in a practical transformer, the primary and secondary coils are each divided into sections, and "sandwiched" between each other on the two legs of the core.

As shown in Fig. 53, not only is there a leakage of flux across from one yoke to the other, but there are also certain lines of force which "leak" out between the turns of the coils on the vertical legs, due to the difference of magnetic potential which exists between the various turns. An approximate expression may also be obtained for this leakage flux, by applying the principles above developed, but it is not necessary to go into this analysis here.

The difference of magnetic potential between the yokes of such a core as shown in Fig. 53 may be many times the resultant magnetomotive force which produces the mutual flux. However, the total leakage flux, due to the high reluctance of its path through the air, is, under practical working conditions, only a small fraction of the mutual flux in the core. Some of the leakage-flux lines which, it should be remembered are always closed loops, just as the main flux lines are closed loops, pass through a part of the iron core, as shown in the figure. However, the portion of the cross-section of the core occupied by these leakage lines is usually so small relative to portion occupied by the mutual or main flux, that the reluctance of the path of the latter may be taken equal to that of a path having the full cross-section of the core.

As noted in Problem 12, Article 90, when a circular ring is covered by a winding whose turns are uniformly spaced, all the flux is confined to the space enclosed by the winding, and there is no flux in the surrounding air (*i.e.*, in the "hole of the doughnut"). This means that there is no difference of magnetic potential between any two points on the ring. That no difference of magnetic potential can exist between two such points follows immediately from the relations above developed. Referring to Fig. 52, let $l$ be the mean circumference of this ring, $\mathfrak{R}$ its total reluctance, and $Ni$ the total ampere-turns of the winding. Let $xl$, where $x$ is a fraction, be the length of any section of this ring, measured along its mean circumference. Then this particular section of the ring threads $xNi$ ampere-turns, and the magnetomotive force corresponding to these ampere-turns is $0.4\pi xNi$. The reluctance of this particular section of the ring is $x\mathfrak{R}$. The flux in the ring is equal to the total magnetomotive force of the winding divided by the total reluctance, namely, $\varphi = \dfrac{0.4\pi Ni}{\mathfrak{R}}$.

Whence the reluctance drop in this section of the ring is

$$(x\Re) \times \left(\frac{0.4\pi \, Ni}{\Re}\right) = 0.4\pi x Ni.$$

Consequently, the *rise* of magnetic potential in any particular section of the ring, due to the ampere-turns threaded by this particular section, is exactly equal to the reluctance *drop* in this section, and, therefore, the resultant difference of magnetic potential between the two ends of this section is zero. Hence there is no difference of magnetic potential between any two points on the surface of the ring, and, therefore, there is no flux established in the air surrounding it. In this deduction it is assumed that the winding on the core forms a perfectly continuous "sheet" of current. In any practical case of a core wound with a coil of insulated wire, a small (usually negligible) amount of flux will leak out between these wires, the lines representing this leakage flux making small "eddies" around the individual turns of the winding.

Problem 15.—The primary coil in Fig. 53 has 640 turns and the secondary coil 64 turns. At a certain instant the current in the primary coil is 1 ampere and the current in the secondary coil is 9.5 amperes. The reluctance of each leg and of each yoke of the iron core to the mutual flux produced by these two currents is 0.000007 oersted. The reluctance of the air to the leakage flux from one yoke to the other is 0.004 oersted.

(a) What is the resultant magnetomotive force, in ampere-turns, tending to produce the mutual flux which links both coils? (b) What is the value of this mutual flux, in maxwells? (c) What is the resultant rise of magnetic potential through the core on which the primary coil is wound? (d) Through the core on which the secondary coil is wound? (e) In what direction is the rise of magnetic potential in each core with respect to the direction of the flux through that core? (f) What is the value of the leakage flux from one yoke to the other?

*Answer.*—(a) 32 ampere-turns. (b) 1,440,000 maxwells. (c) 632 ampere-turns. (d) 616 ampere-turns. (e) The rise of magnetic potential through the primary coil is in the direction of the flux, whereas the rise of magnetic potential through the secondary coil is in the opposite direction to that of the flux. (f) 196,000 maxwells.

**94. Kirchhoff's Laws for the Magnetic Circuit.**—As noted in Article 78, one of the fundamental properties of magnetic flux is that it can be represented by continuous lines which form close loops. As a consequence of this property, it follows that the total flux entering one side of any surface in a magnetic field is always equal to the total flux which leaves the other side of this surface. When a flux which is represented by lines of force whose positive sense is *away from* one side of a surface is consid-

ered as equivalent to a *negative* flux *entering* this side of this surface (see Article 82), an equivalent statement of this fundamental relation is that the algebraic sum of the fluxes entering any surface in a magnetic field is zero. This may be expressed mathematically by the equation

$$\Sigma\varphi = 0 \qquad (28)$$

As already noted, equation (20) is equivalent to the statement that the algebraic sum of the reluctance drops around any closed magnetic circuit is equal to the algebraic sum of the magnetomotive forces established in the various sections of this circuit by the ampere-turns which link them. Or, in mathematical form,

$$\Sigma\Re\varphi = 0.4\pi\Sigma Ni \qquad (28a)$$

where $\Re$ and $\varphi$ are in oersteds and maxwells respectively, and $i$ is in amperes.

These two relations are of identically the same form as Kirchhoff's Laws for a conducting network (see Article 48). They are, therefore, frequently referred to as Kirchhoff's Laws for the magnetic circuit.

The second of the above relations, equation (28a), though useful for rough calculations, can seldom be applied in this particular form to the accurate determination of the distribution of flux in a magnetic field. This is due to the fact that magnetic flux, unlike an electric current, cannot be confined to "insulated" paths of small cross-section (*e.g.*, wires), but fills all the space in the vicinity of the electric currents which produce it. Equation (20), which expresses the same facts in a more general form, is a much more useful relation.

**95. Refraction of Lines of Force at the Surface of a Magnetic Medium.**—A particularly useful relation, in problems involving the determination of the shape and distribution of the lines of force in a magnetic field, is that the line of force through any point in the surface of separation between two bodies of different permeabilities (*e.g.*, iron and air) is always refracted toward the normal in the body of the lesser permeability, provided there is no current in the surface at this point. In fact, due to the high permeability of iron, the lines of force in the air where they enter or leave a piece of iron are practically always perpendicular to its surface, irrespective of their direction within the iron.

This relation may be deduced directly from the two relations: (1) that the magnetic flux which enters one side of a surface is always equal to the flux which leaves the other side of this surface and (2) that the line integral of the magnetizing force around a closed path which does not link a current is always zero. The deduction is as follows:

Referring to Fig. 54, let $dS$ be any elementary area in a magnetic field. Let $B_1$ be the flux density at a point infinitely close to one side of this area (say its $M$ face), and let $B_2$ be the flux density at a point infinitely close to the other side
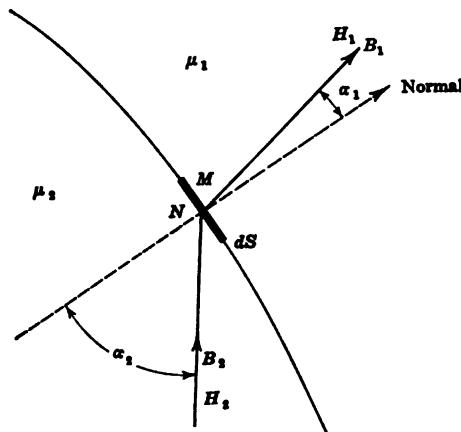


FIG. 54.

($N$ face) of this area. Let $\alpha_1$ be the angle between the direction of the line of force at the first point (*i.e.*, the direction of the flux density $B_1$) and the direction of the normal drawn from this area through this point, and let $\alpha_2$ be the angle between the direction of the line of force at the second point (*i.e.*, the direction of the flux density $B_2$) and the direction of this normal. From equation (9b) the flux which enters the $N$ face of this area is then $(B_2 \cos \alpha_2)dS$, and the flux which leaves its $M$ face is $(B_1 \cos \alpha_1)dS$. Since these two fluxes must be equal,

$$B_1 \cos \alpha_1 = B_2 \cos \alpha_2 \qquad (29)$$

That is, the normal components of the flux densities on the two sides of any surface in a magnetic field are always equal.

Let $H_1 = \dfrac{B_1}{\mu_1}$ and $H_2 = \dfrac{B_2}{\mu_2}$ be the values of the magnetizing

force on the two sides of the given element of surface; $\mu_1$ and $\mu_2$ being the permeabilities of the media on the two sides of this surface. These magnetizing forces, since their directions, like those of the flux densities, are along the line of force through the elementary surface $dS$, likewise make with this surface the angles $\alpha_1$ and $\alpha_2$. Imagine a rectangular loop of infinitesimal width to be drawn around this surface $ds$, in such a manner that one side of this loop is in the medium of permeability $\mu_1$ and the other in medium of permeability $\mu_2$. Since the two sides of this loop are equal in length and its two ends of negligible length, and since by hypothesis there is no current in the surface $dS$, the component of $H_1$ along one side of this loop must be equal to the component of $H_2$ along its other side (see equation (20a)).

These two components are respectively $H_1 \sin \alpha_1$ and $H_2 \sin \alpha_2$, where $\alpha_1$ and $\alpha_2$ are the angles between the directions of the line of force on the two sides of the surface $dS$ and the normal to this surface (see Fig. 54). Whence

$$H_1 \sin \alpha_1 \ = H_2 \sin \alpha_2 \tag{30}$$

or

$$\frac{B_1}{\mu_1} \sin \alpha_1 \ = \frac{B_2}{\mu_2} \sin \alpha_2 \tag{30a}$$

**That is, the tangential components of the magnetizing forces on the two sides of any surface, directly opposite any point in this surface, are always equal,** provided there is no current in the surface at the point in question.

By combining equations (29) and (30a), there is obtained the relation

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\mu_1}{\mu_2} \tag{31}$$

Hence, when $\mu_1$ is less than $\mu_2$, $\alpha_1$ must be less than $\alpha_2$. That is, wherever a line of force passes through the surface of separation between two bodies of different permeabilities, this line is refracted toward the normal in the body of lesser permeability.

This relation may be readily tested experimentally by placing a small magnetic needle near any point in the surface of a magnetized piece of iron. The needle will take a direction which is practically perpendicular to the surface of the iron at this point. Again, when the end of a bar magnet is dipped into a mass of iron filings, the filings which cling to the bar, even those on its

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sides, will be found to stick out practically perpendicular to its surface.

It should be noted, however, that the relation represented by equation (31) holds only *at* the surface of separation between the two bodies in question. A line of force which is perpendicular to such a surface at the point where it intersects this surface may have an entirely different direction at a point even but a small distance from this surface.

Problem 16.—A line of force, which passes out through the side of an iron core (*i.e.*, a leakage line), makes with the normal to the surface of this core within the iron an angle of 85 degrees. The core has a permeability of 2000 c.g.s. electromagnetic units.

(a) What will be the direction of this line of force in the air just outside the core? (b) What will be the relative direction of the equipotential surface through this point and the surface of the core at this point? (c) Draw a diagram showing to scale the direction of the line of force and the direction of the equipotential surface, just within and just outside the core.

*Answer.*—(a) 0.33 degrees from the normal. (b) The equipotential surface will be practically tangent to the surface of the core, since the normals to these two surfaces make an angle of only 3.3 degrees with each other.

## CALCULATION OF FLUX DENSITY

**96. General.**—The various methods of calculation developed in the last chapter are all based on the assumption that the shape of the lines of force in the magnetic fields considered is known. In this chapter will be shown how both the magnitude and direction of the flux density may be calculated for any point in a magnetic field, without making any assumption whatever in regard to the shape of the lines of force. Incidentally, the shape of the lines of force in a number of important cases will be determined.

As pointed out in the last chapter, the introduction of a magnetic substance, such as a piece of iron, into a magnetic field always increases the total magnetic flux in the space which the magnetic substance is made to occupy, and in general changes both the magnitude and direction of the flux density·at every point in its vicinity. Hence, in the calculation of the magnetic flux density at any point in a magnetic field, account must be taken not only of the electric currents which produce the field, but also of the nature, size, shape and position of every magnetic body in the field.

Those cases in which the magnetic field is due entirely to electric currents will be considered first. When there are no magnetic substances present, the flux density and the magnetizing force are numerically equal, when expressed in c.g.s. electromagnetic units, and both have the direction of the line of force through the point under consideration. The formulas developed will be expressed in terms of the flux density, but it should be remembered that a formula, in c. g. s. electromagnetic units, for the value of the flux density at a point at which *there is no magnetic substance* also gives the magnetizing force at this point.

**97. Flux Density Due to a Current in an Elementary Length of its Circuit.**—An electric current can exist only in a closed electric circuit, *i.e.*, every stream line of an electric current is a closed loop, just as every magnetic line of force is a closed loop. (This is true even for the charging current of a condenser, provided both the conduction current and the "displacement cur-

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rent" in the dielectric of the condenser are taken into account; see Article 104). Hence it is impossible to determine by direct experiment the flux density produced by an electric current in a given *portion* of an electric circuit.

However, the results of all known experiments are consistent with the assumption that the current $i$ in each elementary length $dl$ of an electric circuit (see Fig. 55) produces at any point $P$ which is at a distance $x$ from $dl$ a flux density which has the value

$$dB = \mu_s \frac{(i \sin \theta) dl}{x^2} \qquad (1)$$

where $\mu_s$ is the magnetic permeability of free space. When all
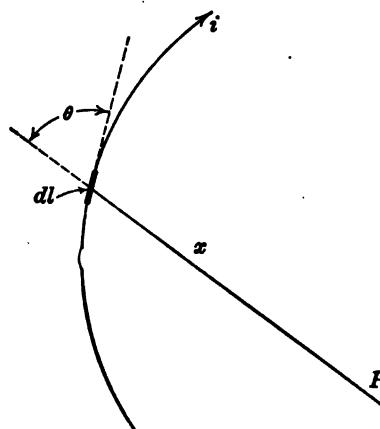


FIG. 55.

quantities are expressed in c.g.s. electromagnetic units $\mu_s = 1$, and this relation may be written

$$B = \frac{(i \sin \theta) dl}{x^2} \qquad (1a)$$

The direction of the flux density $dB$ given by this equation is *perpendicular to the plane determined by the point P and the length dl*, and its positive sense is in the right-handed screw direction with respect to the current $i$. For example, in Fig. 55 the flux density at $P$ due to the current in $dl$ is perpendicular to the plane of the paper, in the direction away from the eye of the reader.

The resultant flux density at any point due to a current in a finite length of an electric circuit, or to a current in a closed electric circuit, is then the *vector sum* of the flux densities at this

point, as calculated from equation (1), for all the elementary lengths which make up this circuit. The resultant flux density due to currents in two or more electric circuits is the *vector sum* of the flux densities due to each current separately.

When there are magnetic substances present, the flux density at the given point calculated from equation (1) in the manner just described will give *that part* of the resultant flux density at this point which is due *directly* to the currents in the electric circuits in the field. To this must be added *vectorially* the flux density at this point due to the magnetized state of all the magnetic bodies which may be in the field (see Article 104).

**98. Flux Density Due to a Current in a Finite Length of a Straight Wire.**—A simple application of equation (1) is to the calculation of the flux density at any point $P$ due to a current in a finite length of a straight wire, in the vicinity of which there is no magnetic material. The formulas derived are particularly useful in connection with problems of electric power transmission, for transmission lines usually consist of wires which are practically straight.



Fig. 56.

Referring to Fig. 56, let $P$ be a point at a perpendicular distance $x = PO$ from a straight wire $BA$ in which the current is $i$ abamperes. Let $dl$ be any elementary length in this wire, measured in the direction of the current, let $l$ be the distance from $O$ to $dl$, let $y$ be the distance from $P$ to $dl$, and let $\theta$ be the angle between $dl$ and the line from $P$ to $dl$. Assume first that the point $P$ is at a distance from the wire large in comparison with the radius of the wire, in which case the wire may be considered as a geometrical line. Note first, from the general rule in regard to the direction of the flux density due to a current in an elementary length, that the flux density at $P$ due to each elementary length of this wire is in the *same* direction, namely, perpendicular to the
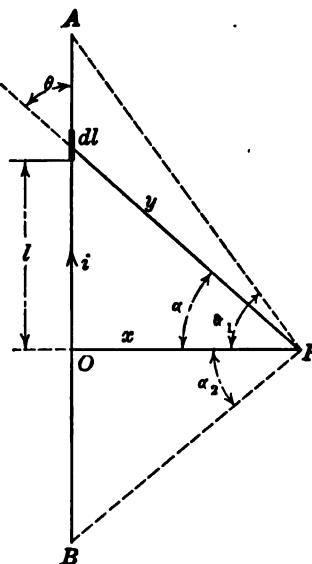
plane through the given point and the wire. Hence, the total flux density at $P$ due to the entire length of the given wire is equal to the *arithmetical* sum of the components due to all the elementary lengths which make up the wire, viz.,

$$B = \int_A^B \frac{(i \sin \theta)dl}{y^2}$$

and its direction is perpendicular to the plane through the given point and the wire, and bears a right-handed screw relation to the direction of the current in the wire.

The simplest way to evaluate this integral is to express the different variables in terms of the angle $\alpha$ between $PO$ and the line from $P$ to $dl$. From the figure,

$$\sin \theta = \cos \alpha, \qquad y = \frac{x}{\cos \alpha}, \qquad l = x \tan \alpha$$

Differentiation of the expression $l = x \tan \alpha$ gives

$$dl = \frac{xd\alpha}{\cos^2 \alpha}$$

Let $\alpha_1$ be the numerical value of the angle between the perpendicular $PO$ and the line from $P$ to one end of the wire, and let $\alpha_2$ be the numerical value of the angle between the perpendicular $PO$ and the line from $P$ to the other end of the wire. The substitution of the above values in the equation for $B$ then gives

$$B = \frac{i}{x}\int_{-\alpha_2}^{\alpha_1} \cos \alpha d\alpha = \frac{i}{x}\left[ \sin \alpha \right]_{-\alpha_2}^{\alpha_1}$$

or

$$B = \frac{i}{x}(\sin \alpha_1 + \sin \alpha_2) \qquad \text{gausses} \qquad (2)$$

In this expression the current $i$ is in abamperes and the distance $x$ is in centimeters.

For any point $P$ on the circumference of a circle concentric with the wire and lying in a plane perpendicular to the wire, the flux density, therefore, has the same value, and its direction is tangent to the circumference at this point. Hence the lines of force due to a current in a straight wire are circles whose planes are perpendicular to the wire and whose centers are in the wire (see Fig. 57), *provided* there is no magnetic material in the vicinity of the wire. The positive sense of these lines bears a right-handed screw relation to the current, as shown in the figure.

The above conclusions are based on the assumption that, with respect to its distance from the point $P$, the wire may be treated as a geometrical line.  In the case of a straight wire of circular *cross-section*, of any size whatever, it may be shown (by applying equation (1) to each of the "filaments" which make up the wire) that the lines of force due to a current which is uniformly distributed over its cross-section are likewise circles, whose planes are perpendicular to the wire and whose centers are on the axis of the wire.  This is true for *all* the lines of force due to this current, including the lines of force which this current may produce *inside* the wire, as well as those outside.

When the conductor has a cross-section of any other shape than that of a circle, the lines of force near and inside the conductor
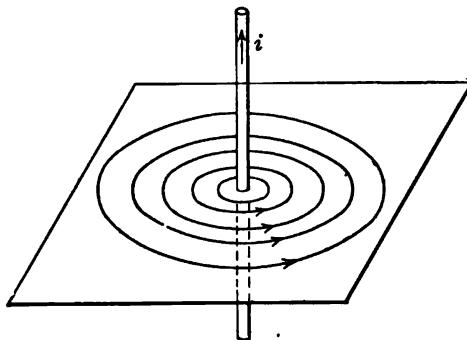


FIG. 57.

will not be circles.  However, for points at a distance from the conductor large in comparison with the greatest linear dimension of its cross-section, the lines of force will be practically circular, and equation (2) will give the value of the flux density to a close approximation, when $x$ is taken as the distance of the point from the center of the wire.

When the distances of the point $P$ from the two ends of the wire are great in comparison with its perpendicular distance $x$ from the wire, the two angles $\alpha_1$ and $\alpha_2$ (see Fig. 56) approach 90 degrees, and their sines become substantially equal to unity. Consequently, from equation (2), the flux density at a point $P$ due to a current of $i$ abamperes in a long straight wire, at a distance of $x$ centimeters from $P$, is

$$B = \frac{2i}{x} \quad \text{gausses} \tag{2a}$$

*provided* the point $P$ is at a distance from each end of the wire great in comparison with its perpendicular distance from the wire, and *also provided*, in general, that its perpendicular distance from the wire is great in comparison with the greatest linear dimension of the cross-section of the wire. In the case of a wire of circular cross-section, however, this second provision is not necessary, except to the extent that the point $P$ must not be *inside* the wire.

By considering a straight wire as made up of an infinite number of parallel filaments of infinitesimal cross-section (each of which may be considered as a geometrical line), it may be deduced directly from equation (1) that the lines of force inside a straight wire of circular cross-section are, as stated above,
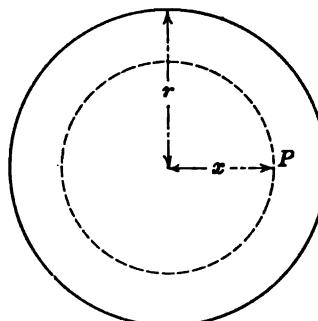


Fig. 58.

circles concentric with the axis of the wire. It is hardly necessary to give this mathematical deduction here, since from symmetry, no other shape for these lines could be expected. It should be noted, however, that for a point inside a wire, the circular loop formed by the line of force through this point links only that proportion of the total current which flows through the area which is bounded by this loop.

Consequently, in the case of a straight wire of circular cross-section, in which the current is uniformly distributed, the flux density at a point $P$ inside the wire (see Fig. 58) is the same as would be produced at this point were it just outside a wire which has a radius $x$ equal to that of the circle formed by the line of force through it, and in which the current is

$$i' = \frac{\pi x^2}{\pi r^2} i$$

where $r$ is the radius of the wire and $i$ the total current in it. Hence, for a point inside a long straight wire in which the current is uniformly distributed, the flux density is, from equation (2a),

$$B = \frac{2xi}{r^2} \qquad \text{gausses} \tag{2b}$$

where $i$ is the total current in the wire in abamperes, $r$ is the radius of the wire in centimeters, and $x$ is the distance of the point considered from the axis of the wire.

Comparing equations (2a) and (2b), it is seen that the flux density at points *inside* a long straight wire of circular cross-section, due solely to a uniformly distributed current in this wire, varies *directly* as the distance of the given point from the axis of the wire, whereas for points *outside* the wire, the flux density varies *inversely* as the distance of this point from the axis of the wire.

All the relations deduced in this article hold only for that component of the total flux density which is due to the current in the particular length of the wire considered. Since the circuit of every electric current forms a closed loop, the resultant flux density, both outside and inside the given length of wire, is that due to all the lengths which make up this closed circuit. Only when the rest of the circuit of which the given straight wire forms a part is at a great distance from this wire, do the relations above deduced hold for the resultant magnetic field around and in the wire, and then only when there is no magnetic material in its vicinity.

It is of interest to note that, on the assumption that the lines of force in the immediate vicinity of a wire are circles, concentric with this wire, equation (2a) may be deduced directly from the general relation (see Article 90) that the line integral of the resultant magnetizing force around any loop which links a current is equal to $4\pi$ times this current (in abamperes). For, if these lines of force are circles, then, from symmetry, the magnetizing force at every point in the circumference of any particular circle has the same value $H$. Let $x$ be the radius of any one of these circles; the length of the circumference of this circle is then $2\pi x$. The line integral of the magnetizing force around this circle is, therefore, $2\pi x H$. Equating this to $4\pi i$, where $i$ is the current linked by this circle, gives for $H$ the value $H = \frac{2i}{x}$.

For a non-magnetic medium, as above assumed, the flux density $B$ is equal to the magnetizing force $H$; whence $B = \dfrac{2i}{x}$, which is the same as equation (2a). Equation (2b) for a point inside the wire may be deduced in the same manner.

It should be noted, however, that the assumption that the resultant lines of force are circles concentric with the wire is correct (approximately) only for points in the vicinity of a wire which is long, straight, of circular cross-section, and in which the current is uniformly distributed, and which is not in the vicinity of a magnetic substance.

In the case of a long straight wire of circular cross-section, surrounded by a concentric tube of magnetic material (*e.g.*, an iron pipe), the lines of force will, from symmetry, be circles concentric with the wire. Hence, from the fundamental relation that the line integral of the magnetizing force around any closed line of force is *always* equal to $4\pi$ times the abampere-turns linked by this closed loop, irrespective of the nature of the material in which this loop is located, it follows that the magnetizing force at each point, whether in the air or in the magnetic material, is, as before, $H = \dfrac{2i}{x}$. In the air the flux density will have this same value, but within the material which forms the magnetic tube, the flux density will be

$$B = \frac{2\mu i}{x} \qquad \text{gausses} \qquad (3)$$

where $\mu$ is the permeability of this material, and $i$ is the current in the wire in abamperes and $x$ is the distance of the point from the center of the wire, in centimeters.

Similarly, *inside* a long straight wire of circular cross-section, which is made of a magnetic material (*e.g.*, iron), the magnetizing force as before is $H = \dfrac{2xi}{r^2}$, and, therefore, the flux density is

$$B = \frac{2\mu xi}{r^2} \qquad \text{gausses} \qquad (3a)$$

where $r$ is the radius of the wire, in centimeters, and the other quantities are as in equation (3).

**Problem 1.**—A wire 5 feet long is bent to form a closed square. What will be the flux density at the center of this square due to a current of 20 amperes in the wire?

*Answer.*—0.594 gauss.

**Problem 2.**—Were the wire in Problem 1 bent to form a closed rectangle whose length is twice its width, what would be the flux density at the center of this rectangle, for the same current as before?

*Answer.*—0.705 gauss.

**Problem 3.**—Were the wire in Problem 1 bent to form a closed rectangle of negligible width (*i.e.*, two sides touching each other, except for the thin layer of insulation between them) what would be the flux density between the two wires, the current being 20 amperes as before? The diameter of the wire is 0.1 inch.

*Answer.*—63.0 gausses.

**Problem 4.**—The two circles in Fig. 59 represent the cross-sections of two long parallel wires, *e.g.*, the two wires of a transmission line. The directions of the current in the two wires are opposite, *e.g.*, toward the eye of the reader in one wire and away from the eye of the reader in the other. Both the wires and the surrounding medium are non-magnetic, and the current is to be assumed uniformly distributed over the cross-section of each wire.

(*a*) Taking the current in each wire as 100 amperes, the radius of each wire
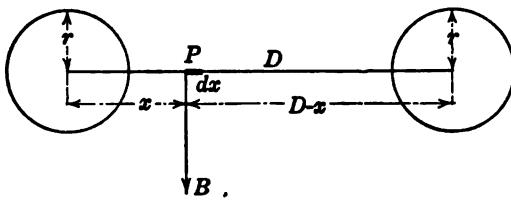


FIG. 59.

as 0.5 inch, and the distance between the centers of the two wires as 6 inches, plot to scale, with the distances from the center of one of the wires as abscissas, the flux density at points on the line joining the centers of the two cross-sections, due (1) to the current in each wire by itself, and (2) to the combined action of both currents. Indicate the point in each wire at which the resultant flux density is zero.

(*b*) Prove that in general, for any point on the line joining the centers of the two cross-sections, but not inside either cross-section, the resultant flux density has the value

$$B = \frac{2i}{x} + \frac{2i}{D - x} \qquad \text{gausses} \tag{4}$$

where $x$ is the distance of this point, in centimeters, from the center of one cross-section, $D$ is the distance, in centimeters, between the centers of the two cross-sections, and $i$ is the current in each wire, in abamperes.

(*c*) Prove that for any point on the line joining the centers of the two cross-sections, and *inside* one cross-section the resultant flux density has the value

$$B = \frac{2xi}{r^2} + \frac{2i}{D - x} \qquad \text{gausses} \tag{4a}$$

where $r$ is the radius of each wire, in centimeters, and the other symbols have the same meaning as in equation (4).

(d) Prove that half the lines of force link one wire and half the other.

(e) Designate, on the curve sheet called for under (a), areas which are proportional to the number of lines of force threaded, in a distance of 1 centimeter measured along the axis of either wire, (1) by the stream line of the current in this wire which is nearest to the other wire, and (2) by the stream line of this current which coincides with the axis of the wire.

(f) Prove that in a distance of 1 centimeter, measured along the axis of either wire, the stream line of the current in this wire which is nearest the other wire is linked by a number of lines of force equal to

$$\varphi' = 2i \log_\epsilon \frac{D - r}{r} \qquad \text{maxwells} \qquad (5)$$

(g) Prove that in a distance of 1 centimeter, measured along the axis of either wire, the stream line of the current which coincides with the axis threads a number of lines of force equal to

$$\varphi'' = 2i \left[ \log_\epsilon \frac{D}{r} + \frac{1}{2} \right] \qquad \text{maxwells} \qquad (5a)$$

(h) Prove that in a distance of 1 centimeter measured along either wire, the *average* number of lines of force threaded by the stream lines of the current is

$$\varphi = 2i \left[ \log_\epsilon \frac{D}{r} + \frac{1}{4} \right] \qquad (5b)$$

**99. Flux Density Expressed in Terms of a Solid Angle.—** Electric circuits are seldom rectilinear, but usually consist of one or more turns. The direct application of equation (1a), Article 97, to the calculation of the flux density due to a current in such a coil, therefore, requires the *vector* addition of an infinite number of infinitesimal components, each of which has a different direction. However, by making use of the conception of the "solid angle" at a given point subtended by a surface, the relation expressed by equation (1a) may be put in a more readily applicable form, as will now be shown.

Consider first an area $A$ in any surface whatever (see Fig. 60), and imagine a sphere of *unit* radius with its center at any point $P$. The area $\omega$ on the surface of this sphere which, if viewed from the center of this sphere, would appear to coincide with the given area $A$, is defined as the "solid angle" at $P$ subtended by the given area $A$.

From this definition it follows that at any point *inside* a *completely* closed surface, the solid angle subtended by this surface is equal to $4\pi$. Therefore, the maximum value of the solid angle at any point is equal to $4\pi$.

When the boundary of the given area $A$ lies in a plane, then for any point *in this plane* and *inside the loop formed by the boundary of this area,* the solid angle subtended by the given area is equal to $2\pi$.
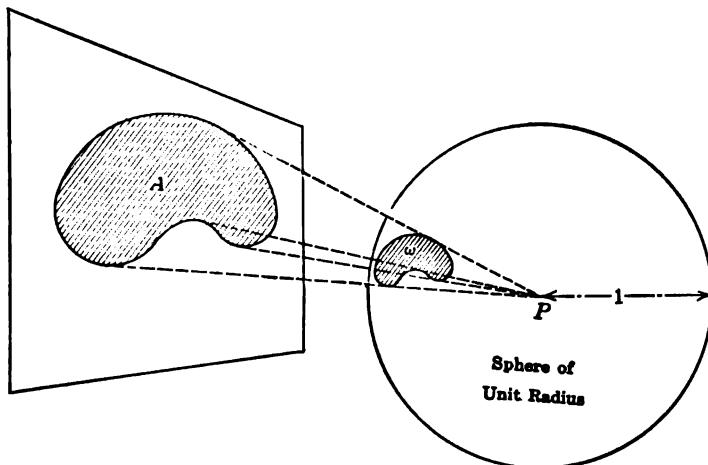


FIG. 60.

Note also that, since the area of a sphere is proportional to the *square* of its radius, the solid angle at the center of a sphere of radius $x$ subtended by an area $S$ in the surface of this sphere is

$$\omega = \frac{S}{x^2} \tag{6}$$

The projection of any elementary area $dS$, on a plane whose
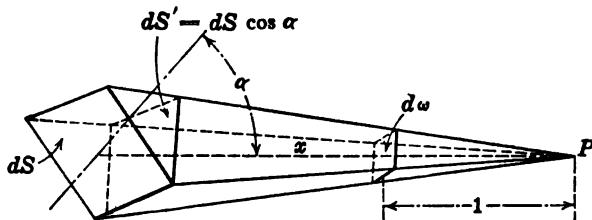


FIG. 61.

normal makes the angle $\alpha$ with the normal to $dS$, is $dS \cos \alpha$. Hence the solid angle at any point $P$ subtended by an *elementary area* $dS$ is

$$dw = \frac{dS \cos \alpha}{x^2} \qquad (6a)$$

where $x$ is the length of the line from $P$ to $dS$ and $\alpha$ is the angle between the direction of this line and the normal to $dS$ (see Fig. 61).

Consider now any curve (or straight line) of fixed length. This curve may either form a closed loop or may be any part of such a loop. Imagine this curve to be given a linear[1] displacement of an infinitesimal amount $dz$ in any direction whatever. The given curve will then trace a surface of definite shape, as
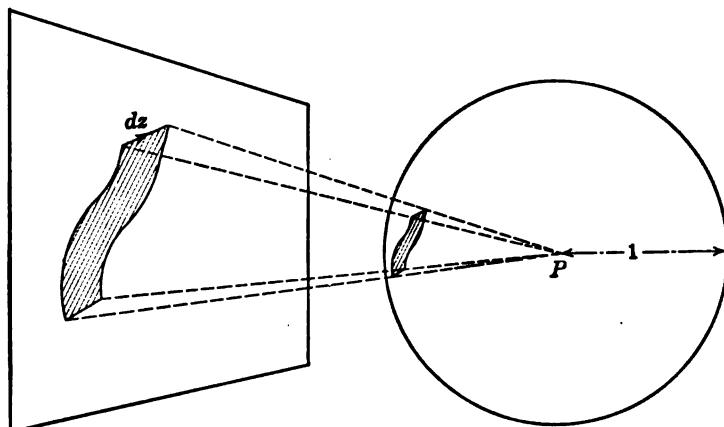


Fig. 62.

shown in Fig. 62. The area of this surface will depend upon the length and shape of the given curve and the amount and direction of the displacement $dz$. At any point $P$ this area will subtend a definite solid angle, depending upon the location of $P$ with respect to this area. This solid angle may be conveniently described as the solid angle at $P$ "swept over" by the given curve when it is given a linear displacement $dz$.

In terms of the ideas just developed, an equivalent statement of the fundamental relation expressed by equation ($1a$) is that the component, in any direction, of the flux density $B$ at any point $P$, due to a current $i$ whose path is a curve of any shape whatever, is equal to ($a$) the inten ity of this current in abamperes multiplied by ($b$) the solid angle at $P$ swept over by this curve when

[1] By the linear displacement of a line or curve is meant that every point of the line or curve is moved the same amount and in the same direction.

it is given an infinitesimal displacement *dz* in this direction, and divided by (*c*) the amount of this displacement. That is, calling *dω* the solid angle at any point *P* swept over by a given portion of an electric circuit when it is given a linear displacement of *dz* centimeters, then the component, in the direction of *dz*, of the flux density *B* at the point *P*, due to a current of *i* abamperes in this portion of the circuit, is

$$B_s = \pm \, i \frac{d\omega}{dz} \qquad \text{gausses} \qquad (7)$$

Which of the two signs is to be used in this expression is determined by the following rule: Holding the thumb and fingers of the *left* hand so that they lie in a plane surface, point the thumb toward the given portion of the circuit under consideration and the fingers in the direction of the current in this portion of the circuit; if the direction of *dz* is such that a line drawn through the hand in the direction of *dz* passes through the hand from the palm to the back, the positive sign is to be used. When the direction of *dz* is such that a line drawn in this direction passes through the hand from the back to the palm, the negative sign is to be used.

For example, imagine a current flowing in a circuit which coincides with the edges of one of the walls of a square room, and let this current be in the right-handed screw direction with respect to the line from the eye to the wall, when the observer stands at any point in the room. The current in each element of this circuit then produces a positive component of flux density in the direction of the line drawn perpendicularly to the wall from the eye of the observer. On the other hand, the current in the edge of the wall which coincides with the ceiling produces vertically *upward* a *negative* component of flux density, while the current in the edge of the wall which coincides with the floor produces, in this same direction, a positive component of flux density. The current in the two vertical edges of the wall produces no vertical component of flux density, for when either of these edges is displaced vertically, neither sweeps over an area.

The component, in any direction, of the total flux density at any point due to a current in a circuit of any shape whatever may then be determined by (*a*) applying equation (7) first to all those portions of this circuit for which the positive sign must be used, then (*b*) to all the other portions of this circuit, and (*c*)

adding these two results *algebraically* (*i.e.*, subtracting their numerical values).

That this method of procedure gives the same results as are obtained by applying equation (1a) directly may be readily shown by considering an elementary length $dl$ of an electric circuit in which the current is $i$ abamperes (see Fig. 63). Let $P$ be any point at a distance $x$ from $dl$, and let the line from $P$ to $dl$ make with the direction of the current in $dl$ the angle $\theta$, as shown in Fig. 63. Let $dl$ be displaced a distance $dz$ normal to the plane determined by $P$ and $dl$, in the direction indicated in the figure. Then $dl$ sweeps over an area equal to $(dl \cdot dz)$. The normal to
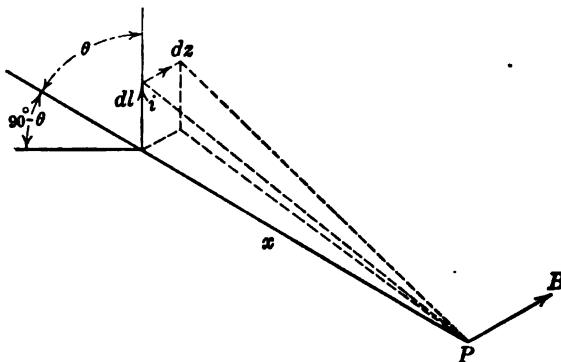


FIG. 63.

this area makes with the line $x$ from $P$ to $dl$ the angle $(90-\theta)$. Hence, from equation (6a), the solid angle swept over by $dl$ is

$$d\omega = \frac{(dl \sin \theta)dz}{x^2}$$

Whence, from equation (7), the flux density at $P$ will have, in the direction of $dz$, the component

$$B_s = + \frac{i(dl \sin \theta)dz}{x^2 dz} = \frac{i(dl \sin \theta)}{x^2}$$

Were $dz$ taken in the plane determined by $P$ and $dl$, the solid angle at $P$ swept over by $dl$ would be zero. Hence $\frac{i(dl \sin \theta)}{x^2}$ is the *resultant* flux density at $P$ due to the current $i$ in $dl$, which relation is identical with that expressed by equation (1a). The direction of this resultant is as indicated in the figure, *i.e.*, normal to the plane determined by $P$ and $dl$ and in the right-handed screw direction with respect to the direction of the current $i$.

Several important practical applications of equation (7) will be given in the following articles.

**Problem 5.**—Prove that at any point $P$ on the axis of a circle the solid angle subtended by the circle is

$$\omega = 2\pi(1 - \cos \theta) \tag{8}$$

where $\theta$ is the angle between the axis of the circle and a line drawn from $P$ to any point in its circumference. (By the axis of a circle is meant the line through its center and which is perpendicular to the plane of the circle.)

**100. Flux Density Due to a Current in a Circular Coil.**— Consider first a closed circuit formed by a single turn of wire bent to form a circle, as shown in Fig. 64. Let $r$ be the radius of this circle, in centimeters, and let $i$ be the current in the wire in abamperes. Let $P_e$ designate the center of the circle.
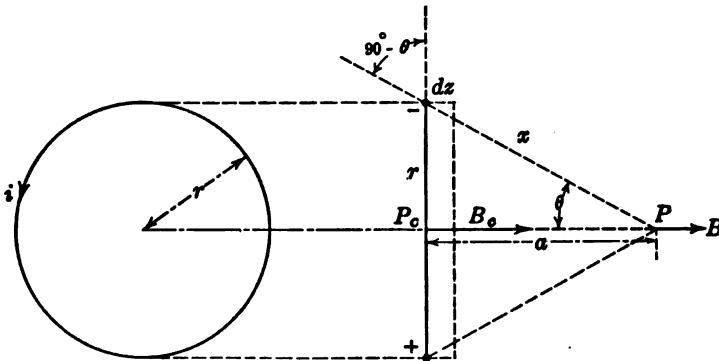


Fig. 64.

Imagine the circle to be given a linear displacement $dz$ in any direction *in its own plane*. No part of its circumference will sweep over a solid angle at $P_o$, since the area swept over by each element of its circumference will be parallel to the line from $P_o$ to this element. Hence the resultant flux density at $P_o$ must be normal to the plane of the circle. (This also holds, for the same reason, for *every* point in the plane of the circle.)

Next, imagine the circle to be given a linear displacement $dz$ normal to its plane, in the direction indicated in Fig. 64. Let the current in this circuit be in the direction indicated in the figure, *i.e.*, toward the eye of the reader at the top (indicated by the minus sign) and away from the eye of the reader at the bottom (indicated by the plus sign). From the rule given in Article 99, the current in each element of the circuit will then produce at $P_o$

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a positive component of flux density in the direction of $dz$, and the resultant flux density will therefore be in this direction and have the value

$$B_c = + i\frac{d\omega}{dz}$$

where $d\omega$ is the solid angle at $P_c$ subtended by a cylinder of radius $r$ and length $dz$, with one end coinciding with the plane of the given circle.

A sphere of radius $r$ drawn about $P_c$ will coincide, for the distance $dz$, with this cylinder, since by hypothesis $dz$ is infinitesimal. Hence this cylinder subtends at $P_c$ the solid angle (see equation (6))

$$d\omega = \frac{2\pi r dz}{r^2} = \frac{2\pi dz}{r}$$

Whence the resultant flux density at $P_o$ due to the current $i$ in the circle of radius $r$ is

$$B_o = \frac{2\pi i}{r} \tag{9}$$

In this formula $i$ is in abamperes and $r$ in centimeters.

Consider next any point on the axis of the given circle, at such a distance from the plane of this circle that the line from $P$ to any point in its circumference makes with this axis an angle $\theta$. Applying the rule given in Article 99 to a linear displacement of this circle in any direction in its *own* plane, will show that corresponding to such a displacement, the current in half the circumference of the circle is in such a direction as to require a $+$ sign in equation (7), whereas for the other half of the circumference a $-$ sign must be used. Moreover, since the two halves of the circumference sweep over equal areas, the total component of the flux density at $P$ in any direction *parallel* to the plane of the circle is zero. Hence the resultant flux density at $P$ is in the direction of the axis of the circle.

To obtain the value of this resultant flux density, imagine the circle to be given a linear displacement $dz$ normal to its plane, in the direction indicated in the figure. Each elementary length $dl$ in the circumference of the circle then sweeps over an area $(dl \cdot dz)$. The area swept over by the entire circumference is then a cylinder of circumference $2\pi r$ and height $dz$. The line of length $x = \frac{r}{\sin\theta}$ from $P$ to any element $dl$ of the circumference of the

given circle makes with the normal to the area swept over by $dl$ the angle $(90 - \theta)$. Hence the solid angle at $P$ swept over by each element of the circumference (see equation $(6a)$) is

$$\frac{(dl \cdot dz) \sin \theta}{x^2} = \frac{dl \cdot dz}{r^2} \sin^3 \theta$$

The total solid angle at $P$ swept over by the entire circumference (whose length is $2\pi r$) is then

$$d\omega = \frac{2\pi r \cdot dz}{r^2} \sin^3 \theta = \frac{2\pi dz}{r} \sin^3 \theta$$

Whence, from equation (7), the resultant flux density at $P$ due to a current of $i$ abamperes in the circle whose radius is $r$ centimeters is

$$B = \frac{2\pi i}{r} \sin^3 \theta \qquad \text{gausses} \qquad (9a)$$

where $\theta$ is the angle between the line from $P$ to any point in the circumference of the circle, *i.e.*, the angle whose tangent is equal to $\left(\dfrac{r}{a}\right)$, where $a$ is the distance of the point $P$ from the plane of the circle.

When a coil has a winding of any number of turns, say $N$, and these turns are so close together that each turn may be considered as sweeping over the same solid angle at any point $P$, when the entire winding is given an elementary linear displacement, the coil is said to have a "concentrated" winding of $N$ turns with respect to this point. From equation (9) it therefore follows that the flux density at the center of a circular coil which has a concentrated winding of $N$ turns, due to a current of $i$ abamperes in this coil, is

$$B_c = \frac{2\pi N i}{r} \qquad \text{gausses} \qquad (9b)$$

where $r$ is the radius of the mean turn of the coil in centimeters.

Similarly, from equation $(9a)$, for any point on the axis of such a coil, at a distance $a$ from the plane of its mean turn, the flux density is

$$B = \frac{2\pi N i}{r} \sin^3 \theta \qquad \text{gausses} \qquad (9c)$$

where $\theta$ is the angle whose tangent is $\left(\dfrac{r}{a}\right)$.

It should be carefully noted that the formulas deduced in this article give the resultant flux density at the point under consideration only when there are **no other currents and no magnetic substances** in its vicinity (see Article 102).

The relations above deduced for the flux density due to a current in a circular coil hold only for *points on the axis* of the coil. It is also possible, by applying equation (7) to any point which is not on the axis of the coil to find an expression for both the magnitude and the direction of the flux density at this point. This expression is by no means simple, and will not be given here.

**Problem 6.**—(a) Plot a curve showing to scale the relation between the flux density at points along the axis of a circular coil and the angle between the axis of the coil and a line from the point considered to the mean circumference of the coil. Call the value of the flux density at the center of the coil 100. Plot $\theta$ as abscissas and $B$ as ordinates.

(b) Plot on the same sheet a curve showing to scale the relation between the flux density at points along the axis of the coil and the ratio $\left(\dfrac{a}{d}\right)$, where $a$ is the distance from the point considered to the center of the coil and $d$ is the diameter of the coil. Plot $\left(\dfrac{a}{d}\right)$ as abscissas and $H$ as ordinates.

**101. Flux Density Due to a Current in a Solenoid.**—A solenoid is a coil of wire of one or more layers, each of which has the form of a long helix. Its cross-section may be of any shape, but it is usually circular. Consider first a single-layer solenoid which has an axial length $l$, a mean radius $r$, and a total of $N$ turns. Let the current in each turn be $i$ abamperes, and let there be no magnetic material in the vicinity of the solenoid. In Fig. 65, the direction of the current, as indicated by the $-$ and $+$ signs, is toward the reader at the top and away from the reader at the bottom.

Let $P$ be any *point within* the solenoid, *i.e.*, inside the space enclosed by the winding, and at a distance less than half the length of the solenoid from the central turn. To find the component of the flux density at $P$ parallel to the axis of the solenoid, imagine the solenoid to be displaced a distance $\triangle z = \dfrac{l}{N}$ parallel to its axis, in the right-handed screw direction with respect to the current, as shown in the figure. The distance $\dfrac{l}{N}$ is equal to the distance between the centers of adjacent turns. Consequently when the solenoid is displaced this distance, the total area swept

over by the path of the current is the surface of a cylinder which has a length equal to the length of the solenoid and a diameter $d$ equal to the diameter of the solenoid (provided the wire which forms the winding may be considered as a geometrical line, *i.e.*, as having a negligible diameter).
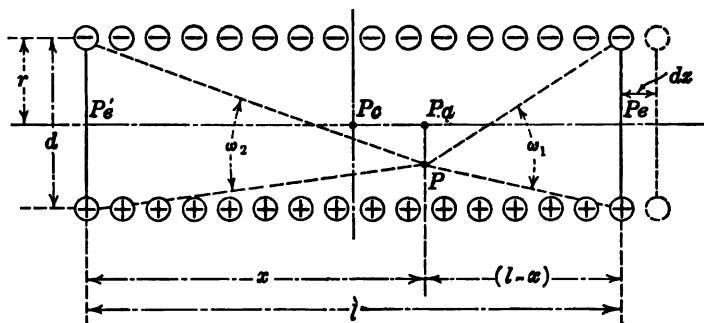


FIG. 65.

Since the total solid angle at a point is equal to $4\pi$, it follows that at the point $P$ this cylindrical surface subtends a solid angle equal to $4\pi$ less the sum of the solid angles at $P$ subtended by the two end faces of the coil, namely the solid angles $\omega_1$ and $\omega_2$ in Fig. 65. Hence, when the solenoid is displaced a distance $\frac{l}{N}$ parallel to its axis, the total solid angle at $P$ swept over by the path of the current in this winding is

$$\Delta\omega = 4\pi - (\omega_1 + \omega_2)$$

Therefore, from equation (7), the component of the flux density at $P$ parallel to the axis of the solenoid is

$$B_s \doteq i \frac{\Delta\omega}{\Delta z} = i \left[ \frac{4\pi - (\omega_1 + \omega_2)}{\dfrac{l}{N}} \right]$$

or

$$B_s = \frac{4\pi N i}{l}\left(1 - \frac{\omega_1 + \omega_2}{4\pi}\right) \qquad \text{gausses} \qquad (10)$$

where $i$ is in abamperes and $l$ in centimeters.

That is, at any point *inside* the solenoid, at a distance from its center (parallel to its axis) less than half its length, the component of the flux density parallel to its axis is equal in gausses to $4\pi$ times the *abampere-turns* of the solenoid *per* centimeter of its

axial length, multiplied by the factor $\left(1 - \dfrac{\omega_1 + \omega_2}{4\pi}\right)$, where $\omega_1$

and $\omega_2$ are the values of the solid angles subtended at this point by the two ends of the solenoid; *provided*, as is assumed throughout this article, there is no magnetic material in the vicinity of the solenoid.

The method followed in deriving equation (10) is mathematically equivalent to the assumption that the solenoid may be considered equivalent to an infinitely thin sheet of conductor which is bent into the form of a cylinder having a length equal to that of the solenoid, and a diameter equal to the mean diameter of its turns, and which carries circumferentially a total current equal to the total ampere-turns of the solenoid. This assumption is not strictly true, and is not even approximately true for points which are very close to, or inside, the wire which forms the turns of the actual solenoid. However, except for such points, equation (10) is substantially an exact relation. An accurate formula for points near or inside the wire is too complicated to be given here.

In an exactly similar manner, it may be shown that equation (10) also holds for any point $P$ inside a solenoid which has a total number of turns $N$ disposed in any number of layers, provided the space *occupied* by the winding is negligible in comparison with the space *enclosed* by the winding. In any actual solenoid, the space occupied by the winding is usually not negligible, but if $\omega_1$ and $\omega_2$ are taken as the solid angles at the two ends subtended by the areas bounded by the end turns of the *mean* layer, equation (10) will give the flux density to a close approximation, except for points near the winding.

It should be noted that in the deduction of equation (10), no assumption whatever is made in regard to the *shape* of the cross-section of the solenoid. The cross-section may be round, square, rectangular, or of any other shape. The only provision in regard to the shape of the cross-section is that it be the same throughout the length of the solenoid, and of the same area throughout.

By applying equation (7) to a linear displacement of a solenoid in a direction *perpendicular* to its axis, it may be shown that, when the solenoid has a length great in comparison with its diameter, the component of the flux density at any point $P$ near its center is practically negligible. This applies to any

point near the center of the solenoid, whether this point be on its axis or not, provided only that it be at a distance from the walls of the solenoid large in comparison with the diameter of the wire which forms the winding.  Hence, within the region near the center of a long solenoid, the lines of force are straight and parallel to its axis, and the value of the resultant flux density at any such point is

$$B = \frac{4\pi Ni}{l}\left(1 - \frac{\omega_1 + \omega_2}{4\pi}\right) \qquad \text{gausses}$$

In the case of a solenoid whose length is great in comparison with its diameter, the sum of the solid angles $\omega_1$ and $\omega_2$ subtended by its ends becomes negligible in comparison with $4\pi$. Consequently, to a close degree of approximation, the resultant flux density at any point inside and near the center of a long solenoid is given by the formula

$$B = \frac{4\pi Ni}{l} \qquad \text{gausses} \tag{10a}$$

where $i$ is in abamperes and $l$ in centimeters.  Note that in this expression $\frac{Ni}{l}$ is equal to the abampere-turns per unit length of the solenoid.

Hence the relation just deduced is equivalent to the statement that the *magnetic field inside and near the center of a long solenoid is uniform and parallel to the axis of the solenoid, and the flux density at each point in this region is equal to $0.4\pi$ times the ampere-turns of the solenoid per unit length.*  This relation, however, is not true for points near the ends of the solenoid, no matter how long it may be, nor is it true for points very close to the winding, even in the region near the center of the solenoid.  It should also be kept in mind that formulas (10) and (10a) are both deduced on the assumption that there are no magnetic substances in the vicinity of the solenoid, and in particular no core of iron or other magnetic material inside it.

An important practical application of equation (10a) is to the calibration of a ballistic galvanometer by the so-called "standard solenoid" method (see Article 110).

From symmetry, for any point $P_a$ *on the axis* of a solenoid *of any length whatever*, provided only that this point be *within* the solenoid (*i.e.*, at a distance from its center less than one-half its length), equation (10) gives the resultant flux density, and

the direction of this resultant density is along the axis. For such points, equation (10) may be put in a more convenient form. Let $x$ be the distance, measured along the axis, of the given point $P_a$ from the "negative" end of the solenoid, *i.e.*, from the end at which the lines of force *enter* the space surrounded by the winding. The distance of this point from the positive end of the solenoid is then $(l - x)$, where $l$ is the length of the solenoid (see Fig. 65). Also let $r$ be the mean radius of the solenoid. Then, from equation (8),

$$\omega_1 = 2\pi \left[ 1 - \frac{l - x}{\sqrt{r^2 + (l - x)^2}} \right]$$

$$\omega_2 = 2\pi \left[ 1 - \frac{x}{\sqrt{r^2 + x^2}} \right]$$

Hence

$$1 - \frac{\omega_1 + \omega_2}{4\pi} = \frac{1}{2} \left[ \frac{x}{\sqrt{r^2 + x^2}} + \frac{l - x}{\sqrt{r^2 + (l - x)^2}} \right]$$

and, therefore, equation (10) gives as the value of the resultant flux density for points on the axis, inside the solenoid, the value

$$B_a = \frac{2\pi N i}{l} \left[ \frac{x}{\sqrt{r^2 + x^2}} + \frac{l - x}{\sqrt{r^2 + (l - x)^2}} \right] \qquad \text{gausses} \quad (10b)$$

In this equation $i$ is in abamperes, and $l$, $r$, and $x$ are in centimeters.

It may also be shown that this equation (10b) holds for *any* point on the axis of the solenoid, whether this point be within or without space enclosed by the winding. Note, however, that when the point is outside the solenoid, either $x$ or $(l-x)$ becomes negative.

For the point $P_c$ which is on the axis and at the center of the solenoid, $x$ is equal to one-half the length of the solenoid. Hence, for this particular point, equation (10b) reduces to

$$B_c = \frac{4\pi N i}{\sqrt{l^2 + d^2}} \qquad \text{gausses} \qquad (10c)$$

where $d$ is the mean diameter of the solenoid, in centimeters.

The ratio of the approximate value of the flux density at the center, as given by the equation $B = \dfrac{4\pi N i}{l}$, to its true value as given by equation (10c), is, therefore,

$$\sqrt{\frac{l^2 + d^2}{l^2}} = \sqrt{1 + \left(\frac{d}{l}\right)^2} = 1 + \frac{1}{2}\left(\frac{d}{l}\right)^2 - \frac{1}{8}\left(\frac{d}{l}\right)^4 + \text{etc.}$$

Hence, neglecting powers of the fraction $\left(\frac{d}{l}\right)$ greater than the second, the per cent. error involved in the approximate formula $B = \frac{4\pi Ni}{l}$ is

$$\text{Per cent. error} = 50 \left(\frac{d}{l}\right)^2 \tag{10d}$$

For example, for a solenoid which has a length equal to 20 times its mean diameter, the error involved in taking for the flux density at its center the value $\frac{4\pi Ni}{l}$, is $50 \times \left(\frac{1}{20}\right)^2 = \frac{1}{8}$ of one per cent.

When the length of the solenoid is negligible in comparison with its diameter, *i.e.*, when the solenoid is a concentrated winding of $N$ turns, equation (10c) reduces to $B_e = \frac{4\pi Ni}{d} = \frac{4\pi Ni}{2r} = \frac{2\pi Ni}{r}$, which is the same as equation (9b).

For the point which is on the axis of the solenoid and in either end face, *i.e.*, for $x = 0$ or $x = l$, equation (10b) gives for the flux density the value

$$B_e = \frac{2\pi Ni}{\sqrt{l^2 + r^2}} \qquad \text{gausses} \tag{10e}$$

For a solenoid which has a length of 10 or more diameters, the flux density at its ends is therefore, with an error of $\frac{1}{8}$ of one per cent. or less, equal to $\frac{2\pi Ni}{l}$. Moreover, since one end face of a long solenoid subtends at any point in its other face a negligible solid angle, it follows from equation (10) that the *component*, along the axis, of the flux density at *any* point in either face of such a solenoid is, to the same degree, of approximation, equal to $\frac{2\pi Ni}{l}$, *i.e.*, to one-half its value at the central cross-section of the solenoid.

Consequently, since the total flux through a given area is equal to the product of this area by the *normal component* of the flux density at this area (see Article 87), it follows that approximately half as many lines of force pass through either end of a long solenoid as pass through its central cross-section. Between the central cross-section and each end, therefore, approximately as many lines pass out through the lateral walls as pass

out through the end surfaces. However, since in the central portion of a long solenoid the lines of force are parallel to its axis,[1] it is only *near the ends* that an appreciable number of lines of force pass out through the lateral walls.

From equation (10b) may be readily calculated the "reluctance drop" through the solenoid. It will be remembered that the reluctance drop from one point to another on the same line of force is equal to the line integral of the magnetizing force along this line from one point to the other (see Article 92). Referring to Fig. 65, let $P_e$ be the point on the axis where it intersects the positive face of the solenoid and $P'_e$ the point on the axis where it intersects the negative face. The direction of the flux through the solenoid is from its negative to its positive face. Hence, since in a non-magnetic medium the magnetizing force is equal to the flux density, the reluctance drop through the solenoid will be the line integral of $B_a$, as given by equation (10b), from $P_e$ to $P'_e$, viz.:

$$(\Re\varphi)_i = \int_o^l B_a dx = \frac{2\pi Ni}{l} \int_o^l \left[ \frac{xdx}{\sqrt{r^2 + x^2}} + \frac{(l-x)dx}{\sqrt{r^2 + (l-x)^2}} \right]$$

$$= \frac{2\pi Ni}{l} \left[ \sqrt{r^2 + x^2} - \sqrt{r^2 + (l-x)^2} \right]_o^l$$

or

$$(\Re\varphi)_i = 4\pi Ni \left[ \sqrt{1 + \left(\frac{r}{l}\right)^2} - \frac{r}{l} \right] \qquad \text{gilberts} \quad (11)$$

This reluctance drop $(\Re\varphi)_i$ may be conveniently referred to as the *internal* reluctance drop through the solenoid.

The *total* reluctance drop around the closed loop formed by any line of force which links all the turns of the solenoid is $\Re\varphi = 4\pi Ni$. Hence the reluctance drop from the positive face of the solenoid to its negative face, in the space *external* to the solenoid, is

$$(\Re\varphi)_e = 4\pi Ni \left[ 1 + \frac{r}{l} - \sqrt{1 + \left(\frac{r}{l}\right)^2} \right] \qquad \text{gilberts} \quad (11a)$$

This external reluctance drop is also equal to the difference of magnetic potential between the two ends of the solenoid.

Only in the case of a very long solenoid, therefore, is the external reluctance drop negligible in comparison with the internal reluctance drop. For example, the external reluctance drop for

[1] Except for the lines which are very close to, or inside, the winding.

a solenoid which has a length of 20 diameters is 2.47 per cent. of the internal reluctance drop.

However, it is interesting to note that the approximate equation $B = \dfrac{4\pi Ni}{l}$ for the flux density at the center of a long solenoid, when the medium inside and around the solenoid is non-magnetic, may be deduced directly from the fundamental relation that the total flux produced by any magnetomotive force is equal to this magnetomotive force divided by the *total* reluctance of the path of the flux produced by it, *provided it is assumed* (1) that the lines of force within the solenoid are straight throughout the length of the solenoid, parallel to the axis, and uniformly distributed and (2) that the reluctance of the path of the flux outside the solenoid is negligible in comparison with the internal reluctance of the solenoid. On these assumptions, the *total* reluctance of the path of the flux would be $\Re = \dfrac{l}{S}$, where $l$ is the length and $S$ is the cross-section of the space enclosed by the solenoid. The total magnetomotive force of the solenoid is $4\pi Ni$, and, therefore, the total flux, on the basis of the two assumptions stated, is $= \dfrac{4\pi Ni}{\Re} = \dfrac{4\pi NiS}{l}$. Therefore, the flux density at the center of the solenoid is $B = \dfrac{\varphi}{S} = \dfrac{4\pi Ni}{l}$.

As shown in the preceding paragraphs, these assumptions lead to results correct within a close degree of approximation when the solenoid is long in comparison with its diameter, for the error introduced by the first of the two assumptions is offset in a large measure by the error introduced by the second assumption. However, in the case of a short solenoid, the errors involved in these two assumptions do not offset each other, and the equation $B = \dfrac{4\pi Ni}{l}$ is not applicable.

**Problem 7.**—A certain air-core solenoid has a length of 50 centimeters, a mean diameter of 10 centimeters, and is wound with 10 turns per centimeter. A current of 5 amperes is established in it. Call the center of this solenoid the origin, and take as abscissas distances measured along the axis from the center.

(a) Plot, as ordinates, the flux density at points along the axis of solenoid. Make actual calculations for $x = 0$, $x = 10$, $x = 15$, $x = 20$, $x = 22.5$, $x = 25$, $x = 27.5$, $x = 30$, $x = 35$, $x = 40$, and $x = 50$. (b) What is the internal reluctance drop through the solenoid? (c) What is the difference

of magnetic potential between the positive and negative ends of the solenoid? (*d*) What percentage of the total magnetomotive force of the solenoid is this difference of magnetic potential? (*e*) What is the reluctance drop through the external space from one end surface to the other? (*f*) Indicate on the curve plotted in (*a*) areas which are respectively proportional to the magnetomotive force of the solenoid, the difference of magnetic potential between its two ends, and the internal reluctance drop.

*Answer.*—(*a*) 61.6, 60.9, 59.2, 53.4, 43.8, 31.2, 15.5, 9.1, 3.2, 1.5 and 0.5 gausses, respectively. (*b*) 2830 gilberts. (*c*) 298 gilberts. (*d*) 9.5 per cent. (*e*) 298 gilberts.

**Problem 8.**—(*a*) Prove that at any point whatever, *outside* an air-core solenoid and not in the space included between the planes through the two end faces of the solenoid, the component of the flux density *parallel to the axis* is

$$B_s = \frac{Ni}{l}(\omega_1 - \omega_2) \qquad \text{gausses} \qquad (12)$$

where $\omega_1$ and $\omega_2$ are the solid angles subtended at this point by the end faces of the solenoid.

(*b*) From equation (12) prove that equation (10*b*) holds for any point on the axis of the solenoid *outside* the space enclosed by the winding and the two end faces of the solenoid.

**Problem 9.**—The winding on a certain solenoid occupies a total axial length of 122.2 centimeters. This winding is in two layers, and the mean diameter of the turns in the two layers is 6.40 centimeters. The total number of turns in the winding is 1263. A second solenoid is placed inside of the first, with its center and axis coinciding with the center and axis of the latter. This second solenoid is wound on a wooden core of circular cross-section, and its winding consists of 407 turns of No. 36 silk-covered wire, in a single layer. The thickness of the silk insulation is 2 mils. The diameter of the wooden core is 4.88 centimeters. A diagram of connections is shown in Fig. 73.

(*a*) How many lines of force pass through each turn[1] of the secondary winding when 1 ampere is established in the primary winding? (*b*) How many linkages are there between these lines of force and the secondary winding? (*c*) When a current of 1 ampere in the primary winding is reversed, what is the total change in the flux linkages of the secondary winding? (*d*) If the secondary winding is connected in series with an external resistance and a ballistic galvanometer, and the total resistance of this circuit is 600 ohms, how many microcoulombs of electricity are discharged through the galvanometer when a current of 1 ampere in the primary winding is reversed?

*Answer.*—(*a*) 245.5 maxwells. (*b*) 99,700 linkages. (*c*) 199,400 linkages. (*d*) 3.32 microcoulombs.

**102. Intensity of Magnetization and Magnetic Susceptibility.**— As has been repeatedly pointed out, the relations thus far de-

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[1] The diameter of each turn is to be taken as the diameter of the circle formed by the axis of the wire.

veloped in this chapter (except equations (3) and (3*a*)) apply only when there are no magnetic substances in the vicinity of the circuits of the electric currents which produce the magnetic field under consideration. The introduction of a magnetic substance into a magnetic field always alters the amount, and in general also the distribution, of the magnetic flux in this field. Experiment shows that this alteration in the field produced by a magnetic substance may be looked upon as due to a property acquired by this substance in virtue of which *it of itself produces a magnetic field, which is superimposed upon the original field*. The degree to which such a substance possesses this property is called its "intensity of magnetization."

In the case of the ferromagnetic substances, it is found that the property thus acquired by it is retained, to a greater or less degree, when the given substance is removed from the field. This fact is usually described by saying the substance in question becomes a permanent magnet. In general, a "magnet" may be defined as any portion of matter in which there is no apparent electric current, but which of itself produces a magnetic field, or alters the magnetic flux in (and in general also around) the space which it is made to occupy.

When the flux produced by a magnet depends primarily upon the presence in its vicinity of an electric current or other magnet, the magnet is said to be an "induced" magnet. As will be shown later, a coil of wire in which an electric current is flowing possesses many of the properties of a magnet, irrespective of whether or not this coil has a magnetic core. A coil is, therefore, frequently called an "electromagnet." When the coil has a magnetic core the term electromagnet is used to include both the core and the coil.

As a measure of the degree to which any portion of a magnetic substance possesses the property of itself producing a magnetic field, *i.e.*, of its "intensity of magnetization," is taken $\frac{1}{4\pi}$ times (*a*) the difference between the resultant flux density $B$ and (*b*) the product of the resultant magnetizing force $H$ in this particular portion of matter multiplied by the permeability of free space. That is, the intensity of magnetization $J$ at any point in a substance is defined by the relation

$$J = \frac{B - \mu_s H}{4\pi} \tag{13}$$

where $B$ is the resultant flux density at this point, $H$ is the resultant magnetizing force at this point, and $\mu_s$ is the permeability of free space.

Since in c.g.s. electromomagnetic units $\mu_s$ is equal to unity, the intensity of magnetization in c.g.s. electromagnetic units is

$$J = \frac{B - H}{4\pi} \qquad (13a)$$

In this system of units the intensity of magnetization may also be written:

$$J = \frac{H}{4\pi}(\mu - 1) \qquad (13b)$$

$$J = \frac{B}{4\pi}\left(\frac{\mu - 1}{\mu}\right) \qquad (13c)$$

where $\mu$ is the permeability of the substance at the point in question.

When the permeability $\mu$ is positive and greater than unity, or when $\mu$ is negative (see Article 111) the intensity of magnetization $J$ has the same algebraic sign as the flux density $B$; see equation (13c). In either of these cases the intensity of magnetization is said to be in the same direction as the flux density, *i.e.*, its direction is taken as the direction of the line of force through the point in question.

When the permeability is positive and less than unity, *i.e.*, for all diamagnetic substances, the intensity of magnetization will have an algebraic sign *opposite* to that of the flux density (see equation (13c)). Hence in such substances the positive sense of the intensity of magnetization at any given point is taken as opposite to that of the line of force through that point. In this case, if $J$ is used to represent the intensity of magnetization in the direction of the lines of force, $J$ is to be interpreted as representing a negative number.

Just as the total magnetic flux through any area in a magnetic field may be conveniently represented by drawing in this field lines each of which has at each point the same direction as that of the flux density at this point, and whose number per unit area normal to their direction is equal to the magnitude of the flux density at this area, so may the magnetization of a substance in a magnetic field be represented by a set of lines *having the same*[1] *direction as the lines of force,* but of such a number that their

[1] Or *opposite* direction, in the case of *diamagnetic* substances.

*density* at each point *is equal to the intensity of magnetization J* at this point. Such lines may be called "lines of magnetization."

Magnetic lines of force, as has been repeatedly pointed out, are always closed loops without ends, and, therefore, the lines of force which enter one side of a surface always pass through this surface and leave its other side. Lines of magnetization, however, need not be, and seldom are, closed loops, but are usually *lines with ends.* For example, in a permanent magnet the lines of force are continuous lines which pass through the magnet from its south pole to its north pole, and which pass through the surrounding space from the north pole to the south pole of the magnet. When the surrounding space is non-magnetic (*e.g.*, air), the intensity of magnetization is zero throughout this space (see equation (13*c*)), and, therefore, there are no lines of magnetization in this region. At each point *inside* the magnet, however, the intensity
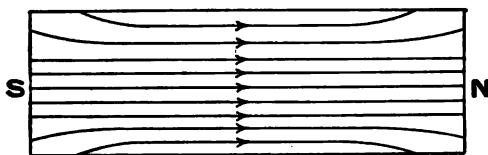


Fig. 66.—Lines of magnetization in a bar magnet.

of magnetization has a definite value, and its direction at each point is from the south to the north pole of the magnet. Hence inside a permanent magnet there are lines of magnetization which originate at its south pole and pass through it, ending at its north pole (see Fig. 66). Compare this figure with Fig. 41, which latter shows the *lines of force* due to a permanent magnet.

From equation (13*b*) it follows that

$$\frac{J}{H} = \frac{\mu - 1}{4\pi} = k \tag{14}$$

where $k$ is a factor which depends upon the nature of the material at the point in question. For a non-magnetic medium this factor $k$ is zero, since for such a medium $\mu = 1$. For a magnetic substance the value of this factor $k$ may be looked upon as the coefficient which expresses ability of this substance to acquire the property of itself producing a magnetic field. This factor $k$ is, therefore, called the "magnetic susceptibility" of the given substance. For all paramagnetic substances the magnetic

susceptibility is a small positive number (practically zero). For diamagnetic substances it is a small negative number. For all ferromagnetic substances the magnetic susceptibility is a relatively large positive number.

**Problem 10.**—In a certain piece of iron a magnetizing force of 10 ampere-turns per inch produces a flux density of 9500 gausses. (*a*) What is the permeability of this iron at this flux density? (*b*) What is its reluctivity? (*c*) What is the intensity of magnetisation of this iron? (*d*) What is its susceptibility?

*Answer.*—(*a*) 1920 c.g.s. electromagnetic units. (*b*) 0.000521 c.g.s. electromagnetic units. (*c*) 756 c.g.s. electromagnetic units. (*d*) 153 c.g.s. electromagnetic units.

**103. Magnetic Poles and Magnetic Pole Strength.**—The chief value of the conception of lines of magnetization is that it leads to an accurate and simple method of defining the "strength"
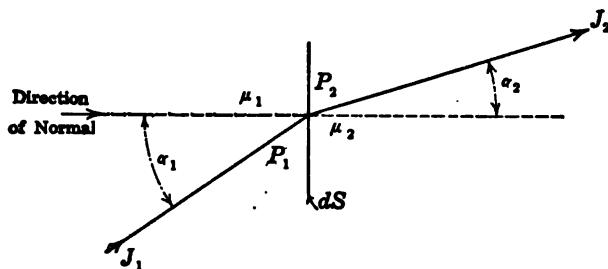


FIG. 67.

of the magnetic pole at any surface in a magnetic field, viz., the *strength of the magnetic pole at any surface in a magnetic field is defined as the number of lines of magnetization which end at this surface.* When more lines of magnetization come up to the given surface than leave it, the pole at this surface is defined as a "positive" or "north," magnetic pole. When more lines of magnetization leave a given surface than come up to it the pole at this surface is defined as a "negative," or "south," magnetic pole.

As a necessary consequence of this definition of the strength of a magnetic pole, it follows that any portion of a magnetic substance which is completely surrounded by a non-magnetic body (*e.g.*, a piece of iron in air) must have equal and opposite magnetic poles. This follows from the fact that every line of magnetization in this portion of matter, if this line has ends at all, must have *two* ends, at one of which there exists a north and at

the other a south pole, and each by definition is of unit strength. Hence every magnet haś equal and opposite magnetic poles.

The above definition of the strength of a magnetic pole may be expressed mathematically as follows:   Referring to Fig. 67, let $dS$ be an elementary area in any surface in the field, and let $\mu_1$ and $\mu_2$ be the permeabilities of the media on the two sides of this area, $\mu_1$ being the permeability of the medium through which the lines of magnetization *come up* to this area.   Let $\alpha_1$ be the angle between the direction of the normal through $dS$ and the direction of the line of force at a point $P_1$ infinitely close to $dS$ in the medium of permeability $\mu_1$, and let $\alpha_2$ be the angle between the direction of this normal and the direction of this same line of force at a point $P_2$ infinitely close to $dS$ but in the medium of permeability $\mu_2$.[1]   The two angles $\alpha_1$ and $\alpha_2$ are to be chosen so that each is less than 90 degrees.   Let $J_1$ and $J_2$ be the intensities of magnetization at the two points $P_1$ and $P_2$, taken as positive when in the same direction as the line of force.

The number of lines of magnetization coming up to the area $dS$ is then $(J_1 \cos \alpha_1)dS$, and the number of lines of magnetization which leave the area $dS$ is $(J_2 \cos \alpha_2)dS$.   Compare with equation (9a) of Article 87 for the number of lines of force through a surface.   Hence the number of lines of magnetization which end on this surface, *i.e.*, the strength of the magnetic pole at this surface, is

$$dm = (J_1 \cos \alpha_1 - J_2 \cos \alpha_2)dS \qquad (15)$$

This relation may also be expressed in terms of the total flux through the surface $dS$ and the permeabilities of the media on the two sides of this surface.   Let $B_1$ and $B_2$ be the flux densities on the two sides of the elementary area $dS$.   The number of lines of force $d\varphi$ through the area $dS$ is equal to $(B_1 \cos \alpha_1)dS$ or to $(B_2 \cos \alpha_2)dS$, viz.,

$$d\varphi = (B_1 \cos \alpha_1)dS = (B_2 \cos \alpha_2)dS$$

Whence, from equation (13c),

$$(J_1 \cos \alpha_1)dS = \frac{d\varphi}{4\pi}\left(1 - \frac{1}{\mu_1}\right)$$

$$(J_2 \cos \alpha_2)dS = \frac{d\varphi}{4\pi}\left(1 - \frac{1}{\mu_2}\right)$$

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[1] When the permeabilities on the two sides of the given surface are different, the angles $\alpha_1$ and $\alpha_2$ must be different, for, as shown in Article 95, a line of force is always refracted at such a surface.

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and, therefore, from equation (15),

$$dm = \frac{d\varphi}{4\pi}\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right) \qquad (15a)$$

In this expression $d\varphi$ represents the flux through the surface in the direction *from* the medium of permeability $\mu_1$ *into* the medium of permeability $\mu_2$. Hence, wherever the lines of magnetic force pass from a medium of high to one of low permeability, $dm$ is positive, *i.e.*, there exists a positive, or north, pole. Wherever the lines of force pass from a medium of low to one of high permeability, $dm$ is negative; *i.e.*, there exists a negative, or south, pole.

The total pole strength on any finite surface $S$ is equal to the algebraic sum of the pole strengths at all the elementary areas which make up this surface. When the permeabilities of the two media separated by the given surface are each constant, equation (15a) gives for the total pole strength

$$m = \frac{\varphi}{4\pi}\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right) \qquad (15b)$$

where $\varphi$ is the total number of lines of force, *i.e.*, the total flux, through this surface.

Certain special cases of these general relations are of particular interest. In the first place, at any surface in a magnetic substance on the two sides of which the permeability has the same value, there exists no magnetic pole. The intensity of magnetization at such a point, however, need not be zero, but may have any value whatever. Again, when none of the lines of force in a magnetic field pass from a medium of one permeability to a medium of a different permeability, there exist no magnetic poles in this field. For example, when a circular iron ring is magnetized by a current in a uniform winding formed by contiguous turns which completely cover the walls of the ring (see Fig. 52), the magnetic field produced by this current is confined entirely to this ring; hence there are no magnetic poles on this ring, no matter how strongly magnetized it may be.

When the given surface is the surface of separation between a magnetic body of permeability $\mu$ and a non-magnetic medium such as air (permeability unity), equation (15b) becomes

$$m = \frac{\varphi}{4\pi}\left(1 - \frac{1}{\mu}\right) \qquad (15c)$$

This relation, therefore, gives the strength of the pole at any portion of the surface at which $\varphi$ lines of force pass from a magnet (permanent or induced) into air or other non-magnetic substance. Where the lines of force pass from the air into the given body equation (15b) gives for $m$ a negative value, which means that any portion of the surface of a magnetic body at which the lines of force *enter* it, there is a negative or south magnetic pole.

When the magnetized body has a high permeability, 1000 or more, say, the term $\frac{1}{\mu}$ is negligible in comparison with unity, and therefore, to a close approximation,

$$m = \frac{\varphi}{4\pi} \tag{15d}$$

or

$$\varphi = 4\pi m \tag{15e}$$

That is, at each unit north pole in the surface of a body of high permeability $4\pi$ lines of force pass out into the air, and at each unit south pole in this surface $4\pi$ lines of force pass into the body from the air. This relation, however, is only an approximation, the actual number of lines of force which pass through a unit pole being

$$\varphi = 4\pi m \left(\frac{\mu}{\mu - 1}\right) \tag{15f}$$

Since every line of force is a closed loop, as many lines of force must come up to a magnetic pole as leave it. For example, from each unit north pole in the end of a permanent magnet there pass out into the air $4\pi$ lines of force (approximately), these lines coming up to the north pole through the body of the magnet. A magnet each of whose poles has a strength numerically equal to $m$ may, therefore, be looked upon as a "sheaf" of $4\pi m$ lines of force (approximately; see Fig. 41).[1]

[1] The term "lines of force through a surface" is used throughout this book to designate the surface integral of the flux density over this surface, viz., $\varphi = \int_S (B \cos \alpha)dS$. This is not the same as the surface integral of the magnetizing force $H$. The surface integral of the latter, namely, the integral $\int_S (H \cos \alpha)dS$, over a *closed* surface containing a pole of strength $m$ is *exactly* equal to $4\pi m$. A unit pole is, therefore, sometimes said to radiate $4\pi$ "lines of force." However, this use of the term "lines of force" to designate the surface integral of the magnetizing force $H$ is extremely confusing, and should be avoided.

From the above definition of the strength of a magnetic pole it follows that it is impossible to have a pole of finite strength at a geometrical point, for this would mean a finite number of lines of force (*i.e.*, a finite magnetic flux) through an infinitely small area, or an infinite flux density. However, under certain conditions (see Article 104) it is permissible, for the purpose of approximate calculations, to consider a pole of finite strength as concentrated in a geometrical point.

When a magnet or any magnetized body is broken in two, and the two parts separated, it is found that equal and opposite magnetic poles appear on the two broken ends, although before the break there exists no apparent pole at the surface of fracture. A rational hypothesis, and one which is fully in accord with the above definition of the strength of a magnetic pole, is to consider the two parts of the magnet which are originally in contact at the given surface as having at this surface poles of equal and opposite strengths, which neutralize each other as long as the two parts are in contact. The strength of the pole at this surface on that part of the magnet through which the lines of magnetic force come up to the surface may be taken, consistently with equation (15), equal to $\int_s (J_1 \cos \alpha_1) dS$, and the pole on the other part of the magnet at this surface may be taken equal to $\int_s (J_2 \cos \alpha_2) dS$.[1] Equation (15) may then be looked upon as defining the *resultant* pole strength at any surface, this resultant pole strength having a value different from zero only when the normal component of the intensity of magnetization on one side of this surface (namely, $J_1 \cos \alpha_1$) differs from the normal component of the intensity of magnetization on the other side of this surface (namely $J_2 \cos \alpha_2$).

The fact that when a magnet is broken into smaller parts, each part manifests the properties of a distinct magnet, leads to the very useful hypothesis that each molecule of a magnet is itself a small magnet. This hypothesis will be considered in greater detail in a subsequent section (Article 114).

Magnetic pole strength is almost invariably expressed in c.g.s. electromagnetic units. All quantities in the various formulas given in this article are to be understood as expressed

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[1] It should be noted, however, that when a magnet is actually broken in two, and the two parts separated, the intensity of magnetization at the surfaces which were orginally in contact in general decreases.

in this same system of units. No particular name has been given to the unit of pole strength, the strength of a given pole being specified simply as so many c.g.s. electromagnetic units.

It is often convenient to speak of the surface at which a magnetic pole exists as being "magnetically charged." The use of this term, however, does not imply that magnetism is an actual thing, like matter or electricity. All that is meant by saying that a surface is magnetically charged is that at this surface lines of magnetic force pass from a medium of one permeability into a medium of different permeability.

**Problem 11.**—A coil of 5 turns of fine wire is connected to a ballistic galvanometer, and is placed around a permanently magnetized iron rod in such a position that the plane of the coil is perpendicular to the axis of the magnet and at a distance of 3 inches from the end of the magnet. When this coil is suddenly pulled off the magnet and completely removed from its field, the first swing of the galvanometer is 20 centimeters. The coil is then put on the magnet again, with its plane 2 inches from the end of the magnet, and when it is suddenly removed, the first swing of the galvanometer is 15 centimeters. The galvanometer, coil and connections have a total resistance of 500 ohms, and a swing of 1 centimeter corresponds to a discharge of 0.4 microcoulomb through the galvanometer.

(a) What is the total magnetic flux through the coil when in its first position on the magnet? (b) When in its second position? (c) What is the total strength of the pole on that section of the magnet included between the two positions of the test coil?

*Answer.*—(a) 80,000 maxwells. (b) 60,000 maxwells. (c) 1591 c.g.s. electromagnetic units.

**104. Magnetizing Force and Flux Density When Magnetic Substances are Present.**—As pointed out in the first part of this chapter, equation (1), Article 96, and the various formulas derived from it for the flux density at any point due to one or more electric currents, holds only when the *entire magnetic field* lies in a *non-magnetic* medium. Similarly, equation (7), which is merely another way of expressing this same relation, and the formulas derived therefrom, hold only when the field is in a non-magnetic medium.

In order to determine the magnitude and direction of the flux density at any point in a magnetic field which lies wholly or partly in a magnetic substance, a more general relation than any thus far stated is necessary. This general relation is that the resultant magnetizing force $H$ at any point in any magnetic field whatever is always equal (1) to the magnetizing force $H_e$ which would be produced at this point by all the electric cur-

rents in this field were there *no* magnetic substances present, plus *vectorially* (2) a magnetizing force $H_m$ which depends only upon the location, strength and distribution of the magnetic poles in the field. In other words, the resultant magnetizing force at any point in a magnetic field may always be looked upon as made up of two components, the first due directly to the electric currents in the field, and the second due solely to the magnetic poles in this field.

The first component of the resultant magnetizing force at any point $P$, namely, the component due directly to the currents in the field, and which may be represented by the symbol $H_c$, is equal to the flux density which these currents would produce at this point were there no magnetic substances present. Its value may always be determined by applying to these currents the
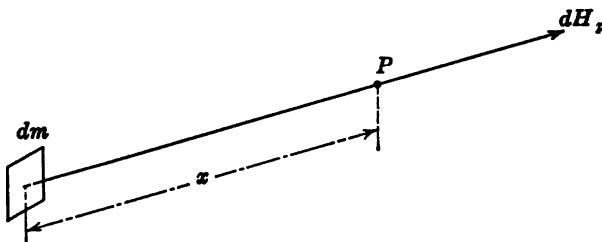


FIG. 68.

general relation expressed by equation (1), taking for $H_c$ the value of $B$ as given by this equation, or by any special formula derived therefrom for any particular case. For example, the value of $H_c$ at any point inside and near the center of a long solenoid is $H_c = \dfrac{4\pi Ni}{l}$, irrespective of whether this solenoid is wound on a non-magnetic core or on a core of iron or any other magnetic material.

The second component of the resultant magnetizing force at any point $P$, namely, the component due to the magnetic poles in the field, which may be represented by the symbol $H_p$, is found by experiment to bear a very simple relation to the strength of these poles and their location with respect to the point in question. This relation is that a magnetic pole which has a strength of $dm$ electromagnetic units, and which is on an elementary surface $dS$ at a distance of $x$ centimeters from the given point $P$ (see Fig. 68), produces at $P$ a magnetizing force of

$$dH_p = \frac{dm}{\mu_s x^2} \qquad \text{gilberts per centimeter} \qquad (16)$$

*in the direction from dm to P.* In this formula $\mu_s$ is the permeability of free space, which in c.g.s. electromagnetic units is 1. Hence the numerical value of $dH_p$ may be written simply

$$dH_p = \frac{dm}{x^2} \qquad \text{gilberts per centimeter} \qquad (16a)$$

This magnetizing force $dH_p$ is entirely independent of the nature of the medium at the point $P$, or between $P$ and the pole $dm$, but depends solely upon the strength of this pole and its distance from the given point $P$.

In equation (16) $dm$ is to be taken as positive when the pole in question is a north pole, and negative when the given pole is a south pole. When $dm$ is so interpreted, $dH_p$ will have a negative value when $m$ is negative. That is, the positive sense of the magnetizing force produced at any point $P$ by a *south* pole is in the direction *from the point to the pole*, whereas the positive sense of the magnetizing force due to a *north* pole is that of a line drawn *from the pole to the point.*

The magnetizing force $H_p$ due to all the magnetic poles in the field is then the vector sum of the magnetizing forces, as calculated from equation (16), for each of the poles in the field. This may be expressed mathematically by the formula

$$H_p = \overline{\sum \frac{dm}{x^2}} \qquad (16b)$$

where the dash (—) over the summation sign is used to indicate the *vector* summation of the quotients $\frac{dm}{x^2}$, the direction of which for the pole on any particular element of surface is that of the line from this element of surface to the point under consideration.

The resultant magnetizing force at any point $P$ due to all the electric currents and all the magnetic poles in the field is then always the vector sum

$$H = \overline{H_e + H_p} \qquad (17)$$

where $H_p$ is given by equation (16b), and $H_e$ is the flux density which would be produced at $P$ by the electric currents in the field were there no magnetic substances present, as calculated from equation (1), or from any special formula derived from this equation, such as equations (2), (4), (7), (9) or (10).

When the permeability at the given point is unity, the resultant flux density $B$ at this point is equal to the resultant magnetizing force $H$ given by equation (17). When the permeability at the point $P$ has the value $\mu$, then the resultant flux density at this point is

$$B = \mu H = \mu \, (\overline{H_e + H_p}) \tag{18}$$

From these relations it is seen that the insertion of a magnetic body into a magnetic field in general changes both the magnetizing force and the flux density at every point in the field. That is, not only are magnetic poles in general "induced" on this particular body, but, when there are other magnetic bodies in the field, the strengths and distribution of the poles on these bodies are likewise altered.

The exact expression for the magnetizing force at any point due to a surface of finite area $S$, magnetically charged with a total pole strength $m$, is given by equation (16b), which involves the consideration of each elementary area $dS$ in this surface and the distance of this elementary area from the given point. However, when the pole at each element of this surface is of the same sign, and the surface $S$ is so far from the given point $P$ that $x$ may be considered as having the same value for every point in this surface, equation (16b) reduces to

$$H_p = \frac{m}{x^2} \tag{19}$$

where $m$ is the total strength of the pole on the entire surface $S$. In using this formula, however, it should always be kept clearly in mind that it is not applicable to a point whose distance $x$ from the surface $S$ is of the same order of magnitude as the dimensions of this surface.

Problem 12.—For points at a distance from the nearer end of a bar magnet not less than half its length, the magnetizing force due to the poles of this magnet may be calculated, to a rough degree of approximation, by assuming its poles to be concentrated at its two ends, in points on its axis, provided the magnet has a length equal to at least 20 times its greatest diameter. (Even under the conditions specified, the error in the calculated value of the magnetizing force may, however, be as much as 25 per cent., depending upon the location of the point with respect to the magnet.)

A given permanent bar magnet, of square cross-section, is 40 centimeters long, and its cross-section is 2 square centimeters. Each pole of the magnet has a strength of 300 c.g.s. electromagnetic units. Draw to scale a longitudinal section of this magnet, and mark its two poles $N$ and $S$ respectively.

Also mark with the letters $P_1$, $P_2$ and $P_3$ respectively (1) the center of the magnet, (2) a point on its axis at a distance of 80 centimeters from its center, and (3) a point at a distance of 80 centimeters from the magnet on a line perpendicular to its axis at its center.

(a) What is the magnitude and direction of the resultant magnetizing force at $P_1$ due to the two poles of the magnet? Designate this magnetizing force by the symbol $H_1$, and show its magnitude and direction by a line through $P_1$ equal in length to $H_1$ and in the proper direction. (b) To what extent will the magnetizing force at any other point in the central cross-section of the magnet differ from $H_1$? (c) What is the magnitude and direction of the resultant magnetizing force at $P_2$ due to the two poles of the magnet? Designate this magnetizing force by $H_2$, and show its magnitude and direction graphically by means of a line drawn at $P_2$. (d) What is the magnitude and direction of the resultant magnetizing force at $P_3$ due to the two poles of the magnet? Designate this magnetizing force by $H_3$, and show its magnitude and direction graphically by means of a line drawn at $P_3$. (e) What is the total number of lines of magnetization through the central cross-section of the magnet? (f) What is the magnitude and direction of the intensity of magnetization at the center of the magnet, assuming these lines uniformly distributed over the central cross-section? Designate this intensity of magnetization by $J_1$, and show its magnitude and direction by means of a line drawn at $P_1$. (g) What is the flux density at the center of the magnet? Designate this flux density $B_1$, and show its magnitude and direction by means of a line drawn at $P_1$. (h) What is the permeability of this magnet at its center? (i) What is the total number of lines of force produced by this magnet?

*Answer.*—(a) 1.5 gilberts per centimeter, parallel to the axis of the magnet, in the direction from its north to its south pole. (b) At every point in the central cross-section of the magnet, the magnetizing force is approximately equal to $H_1$, both in magnitude and direction. (c) 0.0533 gilbert per centimeter, parallel to the axis of the magnet in the direction of a line extended through it from its south to its north pole. (d) 0.0214 gilbert per centimeter, parallel to the axis of the magnet in the direction from its north to its south pole. (e) 300 lines. (f) 150 c.g.s. electromagnetic units. (g) 1885 gausses. (Note particularly that $H_1$ and $B_1$ are in *opposite* directions, but that in such a long magnet as above described, $H_1$ is negligible in comparison with $B_1$.) (h) − 1256 c.g.s. electromagnetic units. (Note particularly the negative sign for $\mu$). (i) 3770 maxwells.

## 105. Distribution of the Magnetic Poles on the Surface of a Magnetic Body.—As shown in the preceding article, both the resultant flux density and the resultant magnetizing force at any point in a magnetic field depend (1) upon the intensities of the electric currents in the field and the shape and dimensions of their circuits, and (2) upon the strength of the magnetic poles in the field, and the distribution of these poles. On the other hand, from the definition of magnetic pole strength (Article

103), the pole strength at any area in the surface of a magnetic body depends in turn upon the resultant flux through this area.

Therefore, just what distribution of poles will be produced when a magnetic substance is placed in a given magnetic field cannot in general be predetermined by any simple calculation, for it is the *resultant* flux density due to the original field and *all*



Original Field

Field after Insertion of Bar of Iron
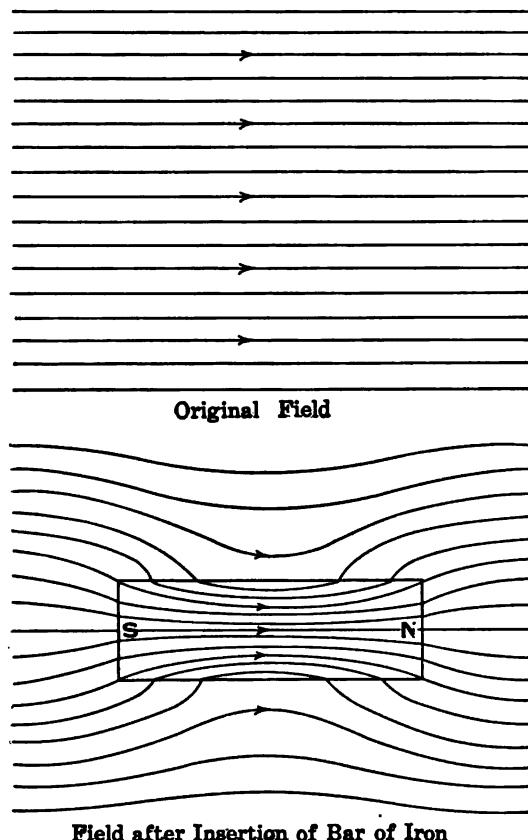
FIG. 69.—Distortion of magnetic field by a piece of iron.

the induced poles which determines the pole strength at any particular point in a surface.

However, one important general conclusion may be drawn from the relations above developed, namely, that when a magnetic body of high permeability is placed in a magnetic field in air or other non-magnetic medium, so that the lines of force pass through

it, entering at one end (or side), as shown in Fig. 69, the poles produced at its surface will be distributed in such a manner that the magnetizing force at any point within the body, *due solely to these* poles, will be approximately equal to, but in the opposite direction, to the magnetizing force originally at this point.

That this is true follows from the fact that, on account of the high permeability of the body, its reluctance to the flux through it is practically negligible in comparison with the reluctance of the rest of the path of this flux, which is through air. Consequently the reluctance drop through this body must be approximately zero in comparison with the reluctance drop through the surrounding air. But the reluctance drop along any portion of a line of force is equal to the integral of the *resultant* magnetizing force along this line (see Article 90). Therefore, the resultant magnetizing force at each point inside the body must be approximately zero. But this resultant magnetizing force is equal to the vector sum of the magnetizing force $H_p$ at this point due to the poles "induced" on this body, and the magnetizing force $H_o$ at this point due to the agents which produce the original field. The only way in which this vector sum can be approximately zero is for $H_p$ to be approximately equal to $H_o$, but in the direction opposite to that of $H_o$.

This general principle, applied to a bar of iron placed in a uniform magnetic field, with its axis parallel to the lines of force (see Fig. 69), shows that the poles of this bar must be distributed not only over its end surfaces, but also over its lateral surfaces near its ends. For were the poles confined solely to its ends, the "demagnetizing" force due to these ends would be very much greater at points inside the magnet in the vicinity of these end surfaces than at points near the center of the magnet, and, therefore, the approximate balance between this demagnetizing force and the magnetizing force due to the original field could not hold throughout the length of the magnet.

Another important fact to keep in mind is that wherever a line of force passes through the surface of a highly permeable body, its direction in the air just outside this surface is practically perpendicular to this surface. (A proof of this statement has already been given; Article 95.) That is, the lines of force in the air in the immediate vicinity of a magnetic pole on the surface

of a highly permeable body are always approximately perpendicular to this surface.

It should also be noted that not only is each element of the external surface of a magnet the seat of a magnetic pole, but that in general each elementary *volume* of the magnet also has a definite pole strength, except in that portion of the magnet through which the lines of force are straight, parallel and uniformly distributed. This follows from the fact that where the lines of force are not straight, parallel and uniformly distributed, the flux density varies from point to point along each line of force, and, therefore, the permeability likewise varies from point to point (see Article 112). Hence on the opposite sides of any area which intersects divergent or convergent lines of force within a magnet, the pole strengths cannot be exactly equal and opposite. Therefore, at such an area there must exist a resultant pole (see the latter part of Article 103). In most practical cases, however, the "internal" poles of a magnet may be neglected.

**106. Solenoid with Magnetic Core.**—In Fig. 70 is shown a longitudinal section of a solenoid with a magnetic core (*e.g.*, an iron or steel rod or bundle of wires) with the axis of the core coinciding with the axis of the solenoid, and with the center of the core at the center of the solenoid. Let $l_s$ be the length of the solenoid, and let $l$ be the length and $d$ the diameter of the core. Let $N$ be the total number of turns in the winding and let $i$ be the current, in abamperes. The $+$ and $-$ signs in the figure indicate the direction of the current in the wire which forms the winding.

When the solenoid has a length of 20 or more diameters, the flux density which would be produced at its center is, from equation (10a) equal to $\dfrac{4\pi N i}{l_s}$. From Article 104, this is also equal to the magnetizing force $H_e$ at this point due *directly* to the current in the solenoid, irrespective of the nature of its core, viz.,

$$H_e = \frac{4\pi N i}{l_s}$$

The direction of this magnetizing force is that of a line drawn through the solenoid in the right-handed screw direction, parallel to its axis, as indicated in the figure. If the core is originally unmagnetized, before the current is established in the solenoid, the resultant lines of force through the core will likewise be in this

direction, and the north and south poles of the core will have the relative positions indicated.

Let $J$ be the intensity of magnetization of the core at its center. From the definition of the pole strength of a magnet, the numerical value of the total strength of each of its pole is $m = J\left(\frac{\pi}{4}d^2\right)$, assuming the intensity of magnetization to have the same value at each point of the central cross-section. When the length of
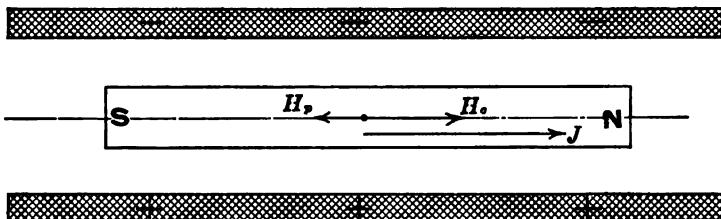


FIG. 70.—Solenoid with iron core.

the core of the solenoid is 20 or more times its diameter, its poles may, to a rough degree of approximation, be considered as located at its ends. On this assumption, the magnetizing force at the center of the core, due to its poles, is, from equation (16b), numerically equal to

$$H_p = 2\,\frac{m}{(0.5l)^2} = 2\pi J \left(\frac{d}{l}\right)^2$$

The direction of this magnetizing force is from the north to the south pole of the core, and, therefore, *opposite* to that of the magnetizing force $H_c$ due to the current in the solenoid. That is, this magnetizing force $H_p$ is a *demagnetizing* force with respect to the magnetizing force due directly to the current.

The resultant magnetizing force at the center of the core is therefore

$$H = H_c - H_p = \frac{4\pi Ni}{l_s} - 2\pi J\left(\frac{d}{l}\right)^2 \tag{20}$$

From equation (13b), $J = \frac{H}{4\pi}\,(\mu - 1)$, where $\mu$ is the permeability of the core. When $\mu$ is large this may be written $J = \frac{\mu H}{4\pi}$, with negligible error. The substitution of this value for $J$ in equation (20) gives for the resultant magnetizing force $H$ at the center of the core the value

$$H = \frac{4\pi Ni}{l_\bullet \left[1 + \frac{1}{2}\mu \left(\frac{d}{l}\right)^2\right]} \tag{20a}$$

The flux density $B$ at the center of the core is, therefore, $\mu$ times this value of $H$, and the total flux through the core is, therefore,

$$\varphi = \frac{4\pi Ni\mu S}{l_\bullet \left[1 + \frac{1}{2}\mu \left(\frac{d}{l}\right)^2\right]} \tag{21}$$

The total reluctance of the entire circuit of this flux, including both the reluctance of the core and the return path of this flux through the air, is then

$$\Re = \frac{4\pi Ni}{\varphi} = \frac{l_\bullet}{\mu S}\left[1 + \frac{1}{2}\mu \left(\frac{d}{l}\right)^2\right] \tag{22}$$

The assumption that the poles of the core are confined to its end surfaces is equivalent to the assumption that all the lines of magnetization terminate at these end surfaces, and, therefore, that these lines, as well as the lines of force, are parallel to the axis of the magnet. On the basis of this assumption, the reluctance of the core alone, exclusive of the return circuit through the air, is

$$\Re_\bullet = \frac{l}{\mu S}$$

(see equation (12) of Article 89).

For example, a core which has a permeability of 1000 and a length equal to 20 times its diameter, the term $\frac{1}{2}\mu \left(\frac{d}{l}\right)^2 = 1.25$. Hence, if the winding on this core has an axial length equal to that of the core, the total reluctance of the complete circuit of the flux is 2.25 times that of the core (*i.e.*, 2.25 times the "internal reluctance" of the electromagnet). Or, the reluctance of the return circuit of this flux through the air outside the solenoid is 1.25 times its "internal" reluctance. Compare with a long air-core solenoid, for which, as shown in Article 101, the external reluctance is practically negligible in comparison with the internal reluctance.

However, it must be carefully noted that equations (20) to (22) are only rough approximations. Actually, the lines of force through the core are not parallel, particularly near its ends, but curve out through its side walls and also pass through the lateral walls of the solenoid. Hence the "effective" value

of the length $l$ in these equations is not the actual length of the magnet but something less than this. Consequently, equation (22) gives too low a value for the total reluctance of the path of the flux. Therefore, if equation (21) is employed to calculate the total flux through the core, the calculated ampere-turns will be too small. The error in the calculated value of the ampere-turns required to produce a given flux may be in error by as much as 50 per cent., or even more, unless the length of the core is very great in comparison with its diameter.

**Problem 13.**—An iron rod, 102 centimeters long and 3.3 centimeters in diameter is uniformly wound with 770 turns of insulated copper wire, forming a winding of the same length as the core. At a flux density of 16,000 lines per square centimeter the permeability of the core is 490 c.g.s. electromagnetic units. Assume the lines of force in the core to be parallel to its axis.

(a) What current is required to establish in this core a flux density of 16,000 lines at its center? (b) What will be the total flux in this core? (c) What will be the total reluctance of the path of this flux? (d) What will be the reluctance of the core itself? (e) What percentage of the reluctance of the core is the reluctance of air portion of the path of this flux? (f) What is the difference of magnetic potential between the two ends of the core? (g) What percentage of the total magnetomotive force of the winding is this potential difference?

*Answer.*—(a) 4.32 amperes. (b) 137,000 maxwells. (c) 0.0305 oersted. (d) 0.0243 oersted. (e) 25.5 per cent. (f) 850 gilberts or 676 ampere-turns. (g) 20.3 per cent.

Note.—For the conditions specified 5.98 amperes were actually required to produce the given flux, as determined experimentally, showing that the error in equation (21), even for a core which has a length of 31 diameters, is 27.8 per cent. This error, as pointed out above, is due to the fact that the lines of force "leak" out through the sides of the magnet. This leakage, as actually measured, and expressed as a percentage of the total flux through the central cross-section of the core, was as follows: In the first 10 centimeters from the center, 1.2 per cent.; in the second 10 centimeters 8.8 per cent.; in the third 10 centimeters 13.7 per cent.; in the fourth 10 centimeters 24.4 per cent.; in the last 11 centimeters 37.5 per cent.; flux which actually came out the end surface of the core 14.4 per cent.

**107. Magnetic Potential Due to a Magnetic Pole.**—As noted in Article 104, the magnetizing force at any point $P$ in space, due solely to a single magnetic pole of strength $m$ located at a point $Q$ at a distance $x$ from this point, is, when all quantities are expressed in c.g.s. electromagnetic units,

$$H = \frac{m}{x^2}$$

and its direction is from $Q$ to $P$ (see Fig. 71). From the definition of drop of magnetic potential given in Article 92, it follows that the drop of magnetic potential $V_{12}$ from any point $P_1$ to any other point $P_2$, due solely to a magnetic pole, is equal to the line integral of the magnetizing force due to this pole, along any path from $P_1$ to $P_2$.

Hence, referring to Fig. 71, the drop of magnetic potential from $P_1$ to $P_2$ due to a pole of strength $m$ at $Q$ is

$$U_{12} = \int_1^2 (H \cos \theta)dl = m \int_1^2 \frac{(\cos \theta)dl}{x^2}$$

where $\theta$ is the angle between the direction of $H$ and $dl$. From Fig. 71 it is evident that $(\cos \theta)dl = dx$. Hence this drop of potential is

$$U_{12} = m \int_1^2 \frac{dx}{x^2} = \frac{m}{x_1} - \frac{m}{x_2}$$
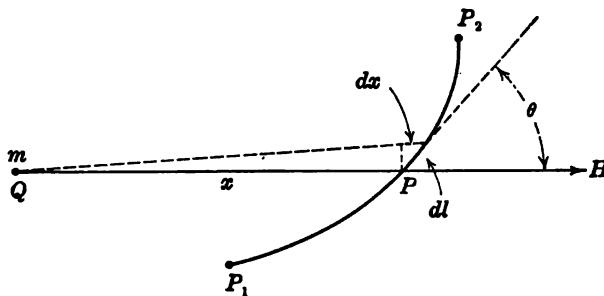


Fig. 71.

where $x_1$ is the distance from $m$ to $P_1$ and $x_2$ is the distance from $m$ to $P_2$.

Since the difference of magnetic potential between the two points is equal to the difference in the values of the quantity $\frac{m}{x}$ for these two points, this quantity may be called the "absolute" magnetic potential due to the pole of strength $m$. That is, the absolute magnetic potential at any point $P$ due to a pole of strength $m$ located at any point $Q$ at a distance $x$ from $P$ is

$$U = \frac{m}{x} \tag{23}$$

It may readily be shown that the total absolute magnetic potential at any point due to any number of magnetic poles, say $N$, is

the *algebraic* sum of the potentials at this point due to all these poles, *i.e.*,

$$U = \sum_{1}^{N} \frac{m}{x} \tag{23a}$$

Since absolute magnetic potential is an *algebraic* quantity, the resultant absolute magnetic potential at any point due to any number of poles is much more readily calculated than the resultant magnetizing force due to these poles. In particular, the absolute magnetic potential $U$ at any point $P$ due to a magnetic pole which is distributed over a surface $S$ is simply the integral of $\dfrac{dm}{x}$ over this surface, where $dm$ is the strength of the pole on any elementary area $dS$ of this surface, and $x$ is the distance of this elementary area from the given point $P$, viz.,

$$U = \int_{s} \frac{dm}{x} \tag{23b}$$

When the absolute magnetic potential at any point $P$ due to any distribution of magnetic poles has been found, then the resultant magnetizing force $H$ at this point may be calculated from the relation that

$$H \cos \theta = -\frac{dU}{dl} \tag{24}$$

where $dl$ is an elementary length taken in this direction, $dU$ is the increase, or rise, in the magnetic potential along this length, and $\theta$ is the angle between the direction of $dl$ and the direction of the resultant magnetizing force $H$ (that is, $H \cos \theta$ is the component of $H$ in the direction of $dl$).

**Problem 14.**—A magnetic pole of total strength $m$ is uniformly distributed over a circular disc (*e.g.*, the end of a round bar) whose radius is $r$ centimeters. Let $\sigma = \dfrac{m}{\pi r^2}$ be the pole strength, or "magnetic charge," per unit area of this disc. Let $P$ be any point on the axis of this disc at a distance $x$ from it. Consider a ring of radius $y$ in this disc, concentric with its center and of a radial width $dy$.

(a) Show that the magnetic potential at $P$ due to the magnetic charge on this elementary ring is

$$dU = \frac{2\pi\sigma y \, dy}{\sqrt{x^2 + y^2}}$$

(b) Show that the total magnetic potential at $P$ due to the entire disc is

$$U = 2\pi\sigma[\sqrt{x^2 + r^2} - x] \tag{25}$$

17

(c) From this last relation show that the magnetizing force at $P$ due to the entire disc is

$$H = 2\pi\sigma\left[1 - \frac{x}{\sqrt{x^2 + r^2}}\right] \tag{26}$$

and that its direction is parallel to the axis of the disc. Note also that when the point $P$ is infinitely close to the disc, the value of the magnetizing force becomes $H = 2\pi\sigma$, which value is independent of the size of the disc.

**Problem 15.**—A circular iron ring is cut through at one point by a hacksaw, forming an air gap $\frac{1}{16}$ inch wide, perpendicular to the circumference of the ring. The cross-section of this ring (*i.e.*, of the iron perpendicular to the circumference of the ring) is 1 square inch. The permeability of the iron is 1000. The ring is permanently magnetized to a flux density of 10,000 lines per square centimeter of its cross-section.

(a) What is the numerical value of the total strength of the magnetic pole on each wall of the air gap, assuming all the lines of force to pass through this gap? (b) What is the numerical value of the pole strength *per square centimeter* of each wall? (c) What is the numerical value of the magnetizing force in the air gap due to each pole by itself? (d) What is the resultant magnetizing force in the air gap due to both poles?

*Answer.*—(a) 5130 c.g.s. electromagnetic units. (b) 796 c.g.s. electromagnetic units. (c) 5000 gilberts per centimeter. (d) 10,000 gilberts per centimeter. (Note that each pole may be looked upon as producing half of the total flux density.)

**108. Flux Density due to a Current in a Toroid.**—A toroid is a uniform winding whose axis is a circle, each unit length of which threads the same number of turns. Such a winding forms a closed ring, like a "doughnut" (see Fig. 52). The space enclosed by the winding (*not* the hole in the "doughnut") is referred to as the core of this winding. This core may be non-magnetic (*e.g.*, air) or it may be magnetic (*i.e.*, an iron ring).

Consider first two equal plane loops symmetrically located with respect to the plane $M$ which bisects the angle between them, as shown in Fig. 72, and let the currents in the two loops have the same values and the same relative direction around the loops, as shown in the figure. Let $P$ be any point on the plane $M$. Imagine the two loops to be given a linear displacement $dz$ parallel to this plane. The solid angle at $P$ swept over by each of these loops will then have the same value, say $d\omega$. The component of the flux density at $P$, in the direction of $dz$, due to the current in the left-hand loop is then, from equation (7), equal to $+\frac{d\omega}{dz}$, and the component of the flux density at $P$, in the di-

rection of $dz$, due to the current in the right-hand loop is $-\dfrac{d\omega}{dz}$. Hence the component, in the plane $M$, of the resultant flux density at $P$ due to the current in both loops is zero. Therefore, the resultant flux density at any point $P$ in the plane $M$ has a direction perpendicular to this plane, as indicated by the arrow at $P$.

This relation, applied to a *uniformly* wound toroid, will show that the resultant flux density at any point, due to the current in all the turns of such a winding, is perpendicular to the plane through this point and the axis of the hole in the "doughnut." Hence every line of force due to this current must be a
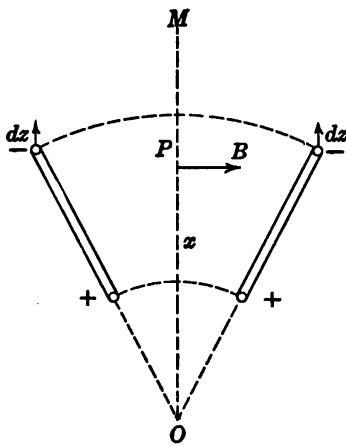


Fɪɢ. 72.

circle concentric with this hole. Moreover, if the solenoid has a magnetic core of the same permeability and same cross-section throughout, there will be no tendency for the lines of magnetization in the core to end on the surface of the core, and, therefore, the core will have no magnetic poles. Hence the resultant lines of force will all be circles even when the core is magnetic.

As a consequence of the fact that the resultant lines of force are circles, it follows that the resultant magnetizing force $H$ will have the same value at every point on any given circle which is concentric with the hole in the "doughnut" (see Fig. 52). Moreover, since no poles are formed on this core, even when this core is magnetic, the magnetizing force at any such point will have the same value, *irrespective of whether the core is magnetic or non-magnetic* (see equation (17), Article 104).

From the fundamental relation that the line integral of the magnetizing force around any closed loop is always equal to $4\pi$ times the abampere-turns threaded by this loop (see Article 90), it follows that the magnetizing force at any point $P$ *inside* the core (whether non-magnetic or magnetic), must have such a value $H$ that $H \times 2\pi x = 4\pi Ni$, where $x$ is the distance of $P$ from the axis of the hole in the "doughnut," $N$ is the total number of turns in the winding, and $i$ is the current in this winding in abamperes. Whence

$$H = \frac{2Ni}{x} \qquad \text{gilberts per centimeter} \qquad (27)$$

Calling $\mu$ the permeability of the core, the flux density at $P$ is

$$B = \mu H = \frac{2\mu Ni}{x} \qquad \text{gausses} \qquad (28)$$

In these formulas $i$ is in abamperes.

These formulas hold only for points inside the closed tube formed by the winding. Outside this space (*e.g.*, in the hole of the doughnut) the magnetizing force and the flux density are zero, since a circle in this outside space threads no ampere-turns, and consequently for each such circle $H \times 2\pi x = 0$. That is, the magnetic field of a uniformly wound toroid is confined entirely to the space inside the closed tube formed by this winding. It should be noted, however, that any practical winding is not actually a closed tube, since there is always a gap between successive turns. Hence for such a winding there is actually a small "leakage" flux in the outside space in the immediate vicinity of the wire which forms the winding. This leakage flux, however, is usually negligible when the turns are close together and uniformly spaced.

It should be particularly noted that all the above deductions in regard to the magnetic field due to a toroid are based on the assumption that the turns which form the winding are uniformly spaced and that the winding covers the entire core. When only a portion of the core is covered by the winding, there will be an appreciable leakage flux through the walls of the uncovered portion of the core, and when this core is magnetic, there will be magnetic poles of appreciable strength formed on these walls. Equations (27) and (28) are then no longer applicable.

When the external and internal radii of the ring formed by the

core of a toroid differ by only a small amount, *i.e.*, when the radial thickness of the ring is small in comparison with its *mean radius a* (see Fig. 52), equations (27) and (28) may be used to calculate the average magnetizing force and average flux density over the cross-section of the core, taking for $x$ the mean radius $a$. Of course this is on the assumption that the core is completely covered by a uniform winding. The average permeability of a sample made in the form of a ring may, therefore, be readily determined experimentally, for the magnetizing force $H$ may be calculated from equation (27), and the flux density $B$ may be measured by means of a secondary coil and ballistic galvanometer (see Article 110).

When the internal and external radii of the core are not substantially equal, the magnetizing force near the inside wall of the core will be greater than it is near the outside wall. Were the permeability independent of the value of the magnetizing force, the flux density would also vary in the same manner. However, the permeability of iron and other ferromagnetic substances is actually a function of the magnetizing force (see Article 111), reaching a maximum value at a definite magnetizing force (dependent upon the nature of the substance). Hence it is entirely possible to have the flux density less near the inside wall of the core than it is near the outside wall. These facts make it necessary to employ, when determining the permeability of a ring sample, a ring whose radial thickness is small in comparison with the internal diameter of the ring (see Problem 12 of Article 90).

Another point of importance in regard to the use of such a ring sample is that, for accurate results, the sample must be a continuous ring. Ring samples are sometimes made up of two half rings, which are, of course, much easier to wind than a single ring. These two half rings when wound are then placed together so that they form a complete ring. However, even when the ends are pressed tightly together, the reluctance of the joints may be appreciable in comparison with the total reluctance of the ring. Or, looked at from another point of view, the poles formed on the two walls of each joint do not exactly coincide, and, therefore, do not completely neutralize each other. Under such conditions, equation (27) will not give the true value of the magnetizing force in the core.

**109. The Earth's Magnetic Field.**—Since a magnetic needle freely suspended at any point near the earth's surface takes up a definite position, pointing approximately north and south (provided there are no other magnets or any electric currents in its vicinity), the region of space in the vicinity of the earth is a magnetic field. In any region in the vicinity of which there are no magnetic substances and no electric currents, the flux density, or "intensity," of the earth's field is practically uniform over a large area, though it changes both in magnitude and direction with the latitude and longitude of the place of observation. The intensity of the earth's field at any point also changes slightly from year to year, and even changes, though by an extremely small amount, from day to day.

The earth's field at any point has in general both a horizontal and a vertical component. Within a certain limited area of the earth's surface in the northern part of Canada, just inside the arctic circle, the direction of the earth's field is vertically downward, *i.e.*, a magnetic needle freely suspended in this region points directly toward the center of the earth. This area is called the "magnetic" north pole of the earth. At a corresponding area in the antarctic zone, approximately diametrically opposite, the direction of the earth's field is vertically upward. This area is called the "magnetic" south pole of the earth. In the vicinity of the "equator" corresponding to these two "magnetic" poles, the earth's field is practically horizontal. The diameter of the earth which connects its two magnetic poles is called the "magnetic axis" of the earth. This axis is not absolute fixed with respect to the axis of rotation of the earth, but is gradually shifting.

The horizontal component of the flux density of the earth's field in the vicinity of Washington is roughly 0.2 gauss, and its vertical component is roughly 0.55 gauss. In any laboratory, however, particularly if it be in the vicinity of electric trolley lines or other circuits carrying heavy electric currents, the magnetic field may differ considerably from this, and will in general vary continually both in magnitude and direction.

Just what causes the earth's magnetic field is not known. For all practical purposes, however, the earth may be thought of as a huge magnet, with its two poles located in the areas above specified. When this point of view is adopted, it should be noted that the magnetic pole in the northern hemisphere is the *south*

pole of this magnet, and the pole in the southern hemisphere is the *north* pole of this magnet. This follows from the fact that the lines of force always enter the south pole of a magnet and leave its north pole; *i.e.*, outside a magnet the direction of the magnetic field in the vicinity of its south pole is *toward* this pole, whereas in the vicinity of its north pole the field has a direction *away from* the pole.

# IX

## HYSTERESIS, MAGNETIC ENERGY AND EDDY CURRENTS

**110. Determination of the Relation Between the Flux Density $B$ and the Magnetizing Force $H$ in a Magnetic Substance.**—The relations deduced in the last chapter lead to a method whereby the variation of the flux density $B$ with the magnetizing force $H$ in a magnetic substance may be readily determined experimentally. The apparatus required is a sample of the material wound with two coils insulated from one another, a
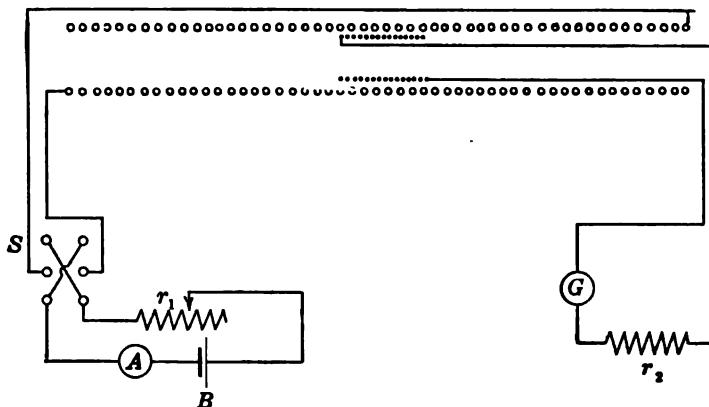


FIG. 73.

ballistic galvanometer, a "standard" solenoid, a source of constant electromotive force, an ammeter, switches, and the necessary rheostats or resistance boxes for varying the strength of the current employed.

A "standard" solenoid is an air-core solenoid having a length of 20 or more diameters, uniformly wound with a known number of turns, say $N_1$, and with a second winding of $N_2$ turns of fine wire wound on a spool which is placed inside the solenoid at its center (see Fig. 73). The two windings are completely insulated from one another. Let the primary winding of the standard solenoid be connected in series with a reversing switch $S$, a

variable resistance $r_1$, a battery $B$, and an ammeter $A$, and let the secondary winding be connected in series with a ballistic galvanometer and a resistance $r_2$.

As pointed out in Article 86, the first swing of a ballistic galvanometer, when a given quantity of electricity $Q$ is discharged through it, is approximately proportional to this quantity of electricity. Any variation of the current in the primary circuit of the solenoid will produce a variation of the magnetic flux through the secondary winding on this solenoid, which in turn will induce an electromotive force in this circuit, and the latter will cause a definite quantity of electricity to flow through the galvanometer and the rest of the secondary circuit.

Let a steady current of $I_1$ amperes be established in the primary winding. Then the flux density at each point in the region occupied by the secondary coil is, from equation (10$a$) of Article 101, equal to

$$B = \frac{0.4\pi N_1 I_1}{l} \qquad \text{gausses}$$

where $l$ is the axial length of the primary winding in centimeters. Let $A_2$ be the area of the mean cross-section of the secondary coil. Then the total flux through each turn of this secondary coil is

$$\varphi_2 = BA_2 = \left(\frac{0.4\pi N_1 A_2}{l}\right) I_1 \qquad \text{maxwells}$$

From equation (7$b$) of Article 86 the reversal of a flux $\varphi_2$ through the secondary winding will cause

$$Q = \left(\frac{2}{10^8 R_2}\right) N_2 \varphi_2 \qquad \text{coulombs}$$

of electricity to flow through the galvanometer, where $R_2$ is the total resistance of the secondary winding, in ohms, including the secondary coil, the galvanometer, the connecting wires and the resistance $r_2$. This discharge of electricity through the galvanometer will cause its moving element to swing out momentarily from its zero, or equilibrium, position. Hence the reversal of a given current in the primary coil, causing a reversal of flux through the secondary, will produce a definite momentary deflection of the galvanometer.

By noting this deflection and the value of the current which is reversed, a "calibration curve" for the given galvanometer

may be plotted, with deflections $D$ as abscissas and flux linkages $\lambda$ as ordinates, these flux linkages being calculated from the formula

$$\lambda = N_2\varphi_2 = \left(\frac{0.4\pi N_1 N_2 A_2}{l}\right) I_1 \qquad \text{maxwells} \qquad (1)$$

This calibration curve should be practically a straight line. It would be exactly a straight line were the galvanometer ideally perfect.

The given galvanometer may then be used to measure the change in the flux linking *any* circuit connected to its terminals, provided the total resistance of this circuit (including the galvanometer) is kept the same as during the calibration, and that the number of turns, say $N'_2$, linked by this flux is known. For example, suppose that the deflection due to a change in flux is $D$, and that this deflection corresponds to $\lambda$ flux linkages, as read from the calibration curve. Then the change in flux through the coil of $N'_2$ turns is

$$\Delta\varphi_2 = \frac{\lambda}{N'_2} \qquad \text{maxwells} \qquad (2)$$

provided each turn is linked by the same flux.

When the galvanometer has been properly calibrated, the standard solenoid is replaced by the sample to be tested, the primary winding on this sample being connected to the middle points of the reversing switch $S$ and the secondary winding to the ballistic galvanometer. The total resistance of the secondary circuit, however, must be kept the same as before. A convenient way of avoiding any adjustment of the resistance of the secondary circuit is to keep the secondary winding of the solenoid and the secondary winding on the sample permanently connected in series, both during the calibration of the galvanometer and during the subsequent tests.

The next step is to thoroughly demagnetize the sample to be tested. Experiment shows that this may be done by establishing a very strong magnetic field in it, and then, while continually reversing this field, gradually reducing its strength. This may readily be done by establishing a large current in the primary winding on the sample, and then, while rapidly reversing the switch $S$, reducing the primary current by inserting more and more resistance in the primary circuit.

The secondary coil on the sample merely gives a means whereby

the total flux through it may be measured. In order to calculate the flux density in the sample from the measured value of this flux, it is necessary that the sample be of such a form that the flux density will be uniform over its cross-section. This condition is realized to a fair degree of approximation (see Article 108) when the sample is made in the form of a circular ring of uniform cross-section, with its axial width small in comparison with the internal radius of the ring (see Fig. 52), provided the primary, or magnetizing coil, is uniformly wound over the entire ring. When these conditions obtain, the change in the flux density in the ring corresponding to a given change in the current in the primary coil will be

$$\Delta B = \frac{\Delta \varphi_2}{A'_2} = \frac{\lambda}{N'_2 A'_2} \qquad \text{gausses} \qquad (3)$$

where $\lambda$ is the flux linkages corresponding to the observed deflection of the galvanometer, $N'_2$ is the number of turns in the secondary winding on the sample, and $A'_2$ is the cross-section of the sample, in square centimeters.

When such a ring sample is used, the magnetizing force established in the ring by a current of $I_1$ *amperes* in its primary winding is, from equation (27) of Article 108, equal to

$$H = \left(\frac{0.4 N'_1}{d}\right) I_1 \qquad \text{gilberts per centimeter} \qquad (4)$$

where $d$ is the mean diameter of the ring in centimeters and $N'_1$ is the number of turns in the primary winding.

In order to obtain the relation between the flux density $B$ and the magnetizing force $H$ in the sample under test, the procedure is as follows. Starting with the iron completely demagnetized as above described, and with a high resistance in the primary circuit, close the reversing switch in the upper position, say, and note the steady value $I'_1$ of the primary current and the first swing $D$ of the galvanometer. From the calibration curve of the galvanometer read the corresponding value of the flux linkages $\lambda$. From equation (3) calculate the increase $(\Delta B)'$ in the value of the flux density in the core, corresponding to the increase in the primary current from zero to $I'_1$. From equation (4) calculate the value of the magnetizing force $H'$ corresponding to the current $I'_1$.

Next, short-circuit a portion of the external resistance in the primary, so that the primary current increases to a new value,

$I''_1$ say, and again note the first swing of the galvanometer. (Of course, the moving element of the galvanometer should always be at rest before any change is made to cause it to swing.) From equation (3) calculate the *increases* $(\Delta B)''$ in the flux density corresponding to the *increase* of the primary current from $I'_1$ to $I''_1$. From equation (4) calculate the value of the magnetizing force $H''$ corresponding to the current $I''_1$. The increase in flux density $(\Delta B)''$ added to $(\Delta B)'$ will give the flux density $B''$ corresponding to the magnetizing force $H''$.

By proceeding step by step in the manner just described, the flux density corresponding to successive values of the magnetizing force up to any value $H_m$ may readily be determined. In a similar manner, when this maximum value has been reached, the flux density corresponding to successively smaller values of the magnetizing force may be determined, by increasing the resistance in the primary circuit step by step, and finally opening the primary circuit (infinite resistance). The reversing switch may then be closed in the opposite direction, and, by the same procedure, the flux density corresponding to successively greater values of the magnetizing force in the "negative" direction may be determined, until the same numerical value $H_m$ is reached, and then for successively smaller values of $H$ down to zero again. The current in the primary winding may then again be reversed and increased in steps until the magnetizing force again reaches the value $H_m$ in the positive direction, and so on for any number of cycles.

Instead of using a ring sample, such as above described, a straight bar, rod or bundle of strips may be employed as the test sample. The magnetizing winding and the secondary winding can be much more readily put on such a sample than on a ring sample, but, as noted in Article 106, the demagnetizing force due to its poles can be determined only to a rough degree of approximation. However, by providing a heavy soft iron yoke for the return circuit of the flux, and by placing at the ends of the sample suitable compensating windings, a degree of precision equal to that obtainable from a ring sample may be secured (see the article on *Magnetic Testing* in Pender's *Handbook for Electrical Engineers*).

**111. Magnetic Hysteresis.**—The results of such a test on a typical sample of iron are shown in Fig. 74. From this curve it

is seen that at first, when the magnetizing force is small, the flux density increases relatively slowly as the magnetizing force increases. When the magnetizing force is increased further, the flux density increases very rapidly; when the magnetizing force
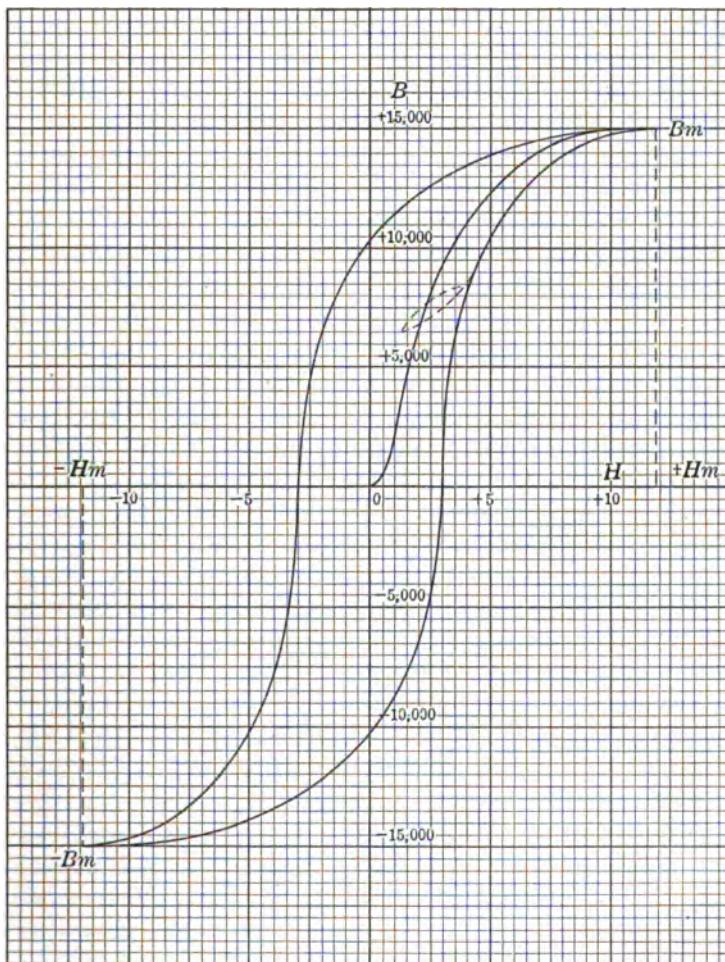


FIG. 74.—Typical hysteresis loop.

becomes still greater, the flux density increases more and more slowly, and finally any further increase in the magnetizing force produces only a comparatively slight change in the flux density,

When the magnetizing force is then reduced, the flux density.

instead of returning to the same values it had for the increasing values of the magnetizing force, decreases less rapidly than it increased; that is, the decreasing values of the flux density lag behind the values corresponding to an increasing magnetizing force. This phenomenon has, therefore, been given the name "hysteresis," from a Greek word meaning "lagging behind."

When the magnetizing force is reduced to zero, the flux density still has a relatively large value; this value is called the "residual magnetism" of the sample. To reduce the flux density to zero, the magnetizing force has to be reversed and increased to a value corresponding to the abscissa of the point where the left-hand branch of the curve cuts the horizontal axis. This value of the magnetizing force is called the "coercive force."

When the magnetizing force is still further increased in the negative direction, the flux density reverses in direction, increasing very rapidly at first and then more slowly. When the magnetizing force reaches the same value in the negative direction as its maximum value in the positive direction, the flux density likewise reaches a maximum in the negative direction equal to the maximum value it had in the positive direction. When the magnetizing force is now reduced to zero and then increased again in the positive direction to its original maximum value, the flux density passes through the series of values represented by the right-hand branch of the curve, which is symmetrical with the left-hand branch. The closed curve formed by these two branches is called the "hysteresis loop."

It should be noted that when the sample is originally unmagnetized, the curve giving the relation between the flux density and the magnetizing force, when the magnetizing force is first increased from zero up to the maximum value, does not coincide with either branch of the hysteresis loop, but is a curve which lies between the two branches of the loop. After the completion of a cycle of changes in the magnetizing force from a positive maximum to an equal negative maximum, and then back again to the positive maximum, the magnetizing force may then be reversed back and forth any number of times between these equal positive and negative maximum values, and the relation between the flux density and the magnetizing force will be the same for each cycle of changes in the magnetizing force as for the first cycle, provided each cycle is performed in the same interval of time.

Experiment shows, however, that when the cycle of changes in flux density is performed quickly, say in 1 second or less, the hysteresis loop, though retaining the same general form, may differ appreciably in its dimensions from the loop corresponding to a cycle which is performed slowly, as, for example, by the step-by-step method just described. This is due to the fact that when a given magnetizing force is applied, the flux density does not immediately come to a constant stable value, but gradually "creeps up" to a stable value. In fact, the flux density may not become absolutely constant for several minutes, or, in the case of very weak fields, for several years, after the magnetizing force is established, depending upon the nature of the material. For example, steel structures in the earth's magnetic field show immediately after erection only a small part of the magnetization which they finally assume after many years.

Since the alternating fluxes employed in practice usually alternate very rapidly, passing through a complete cycle of values in $\frac{1}{25}$ second or less, the hysteresis loop corresponding to a rapid change in the flux density is more important, from a practical standpoint, than the loop obtained by the slow step-by-step method above described. Such a loop may be obtained experimentally by starting each time with the current in the magnetizing winding at the value corresponding to the *tip* of the hysteresis loop (*i.e.*, corresponding to the magnetizing force $H_m$ in Fig. 74), and reducing this current by successively greater steps, always returning to the maximum value after each step. The corresponding deflections of the ballistic galvanometer will then be proportional to the (practically) instantaneous change $\Delta B$ in the flux density *from its maximum value* $B_m$ to the value which corresponds to the particular value of $H$ to which the magnetizing force is reduced. The flux density corresponding to this value of $H$ is then $B = B_m - \Delta B$. The maximum flux density $B_m$ is found by reversing the current, in which case $\Delta B = 2B_m$.

The curve in Fig. 74 gives for any value of the flux density between zero and the maximum value $B_m$ *three* possible values of the magnetizing force. In fact, it is possible, by first increasing and then decreasing the magnetizing force by suitable amounts (see the dotted loop in Fig. 74), to establish for any given final value of the magnetizing force a flux density of any value between the two limits given by the two branches of the

hysteresis loop. For example, at a magnetizing force of 4 gilberts per centimeter the flux density in the particular sample to which Fig. 74 applies may have any value between 8000 and 13,500. Again, when the sample is originally magnetized, so that the initial flux density is not zero, the entire hysteresis loop is shifted above or below the axis of *H*, and is in general distorted, so that it is no longer symmetrical.

As already noted, when the magnetizing force in a piece of iron or other magnetic substance is reduced to zero, the flux density does not in general become zero, but may retain a relatively large value, usually referred to as the "residual magnetism." This residual flux density is a measure of the degree to which the body can be permanently magnetized. In this connection it should be noted that in the case of a bar or rod which is magnetized by a current in a coil wrapped around it, the *resultant* magnetizing force does not become zero when the current is interrupted, but falls to a negative value, due to the effect of the poles at the ends of the bar. Hence the permeability of a permanently magnetized bar is always negative. Only in the particular case of a uniformly wound toroid does the resultant magnetizing force become zero when the current in the winding is interrupted. The core of the toroid is then permanently magnetized, but has no magnetic poles. The permeability of the core under these conditions is infinite, *i.e.*, zero magnetizing force and a finite flux density.

The "retentiveness" of a magnetic substance, *i.e.*, its property of remaining permanently magnetized, depends both upon its composition and its previous heat treatment. Soft iron has only a slight degree of retentiveness, whereas hard steel will retain a relatively high degree of magnetism. Jarring a magnet, or any violent mechanical shock, tends to reduce its magnetism, this effect being much more pronounced for soft iron than for hard steel.

**112. B-H Curves.**—Since for any given value of the flux density there are in general an infinite number of possible values of the magnetizing force, between two definite limits, the permeability of a magnetic substance is not a definite quantity, even for a given value of the flux density, but is dependent upon the manner in which this flux density is established.

Starting with a sample of iron completely demagnetized, the step-by-step method of determining the relation between the

flux density $B$ and the magnetizing force $H$ gives a perfectly definite curve, provided the magnetizing current is not decreased at any time during the test. This curve is sometimes referred to as the "rising" magnetic characteristic.
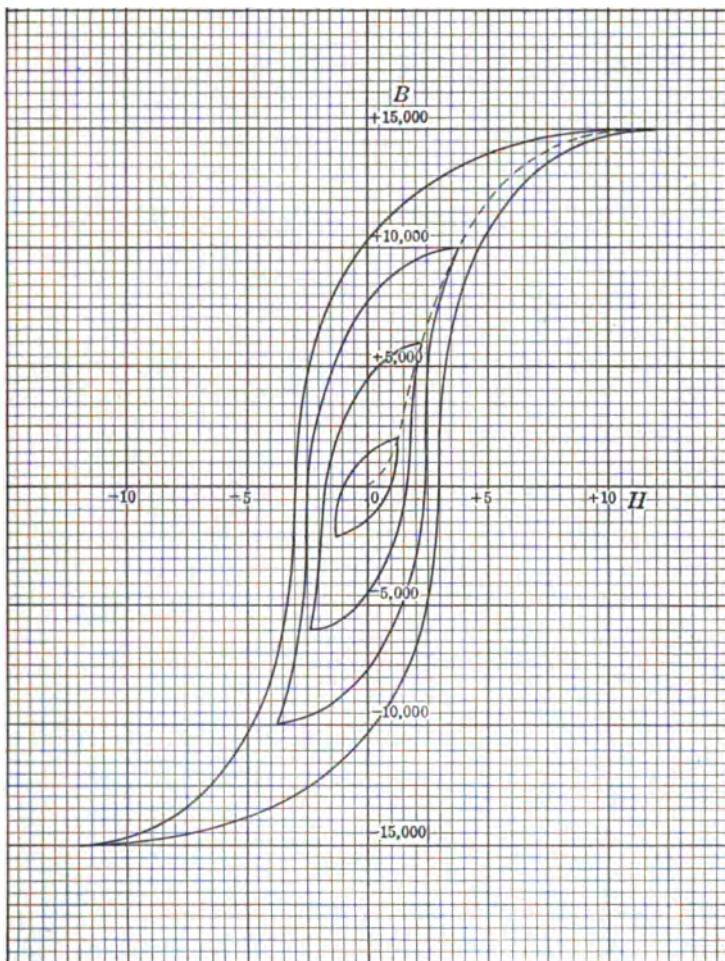


Fig. 75.

Another single-valued relationship between $B$ and $H$ may be obtained by the so-called "method of reversals." In this method, the sample is first demagnetized and a magnetizing current of any given value is established in the primary winding. The

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ballistic galvanometer is then connected to the secondary on the test sample, and its deflection is noted when this current is reversed. This deflection is proportional to twice the flux den-
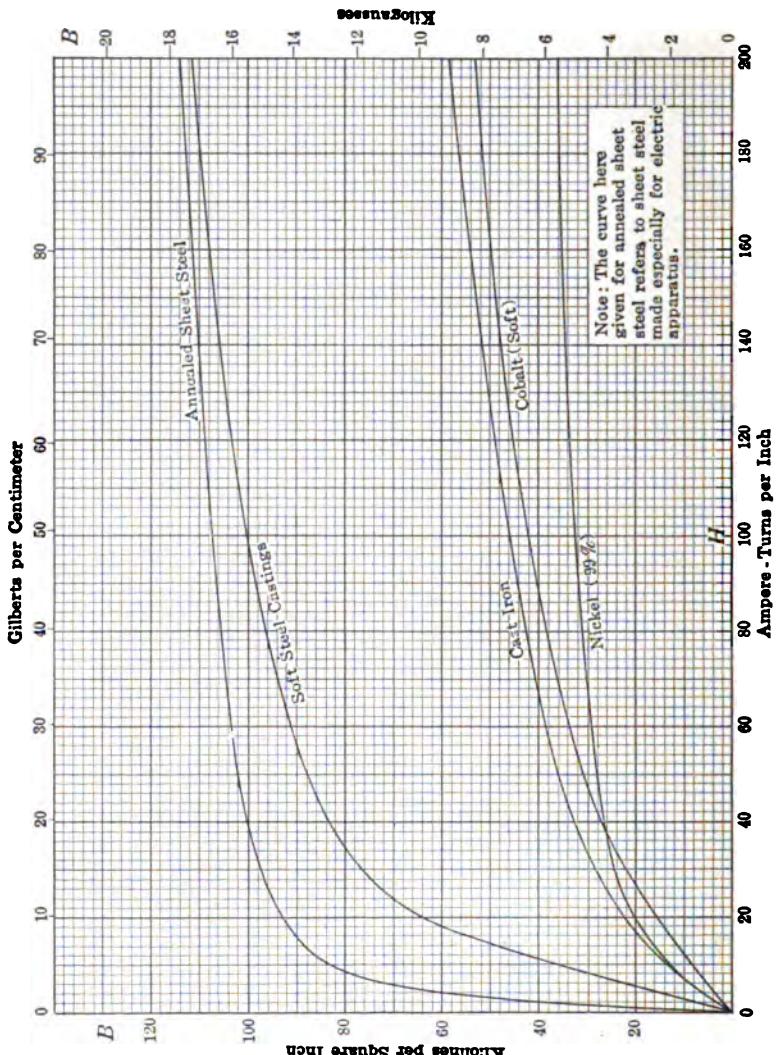


Fig. 76.—Typical *B-H* curves.

sity corresponding to the particular value of the magnetizing current employed. The magnetizing current is then increased to a higher value, reversed several times, and another set of observations is taken, corresponding to the final reversal, and

so on, for the desired range of flux densities.  The flux densities
as thus determined, plotted against the magnetizing force (as
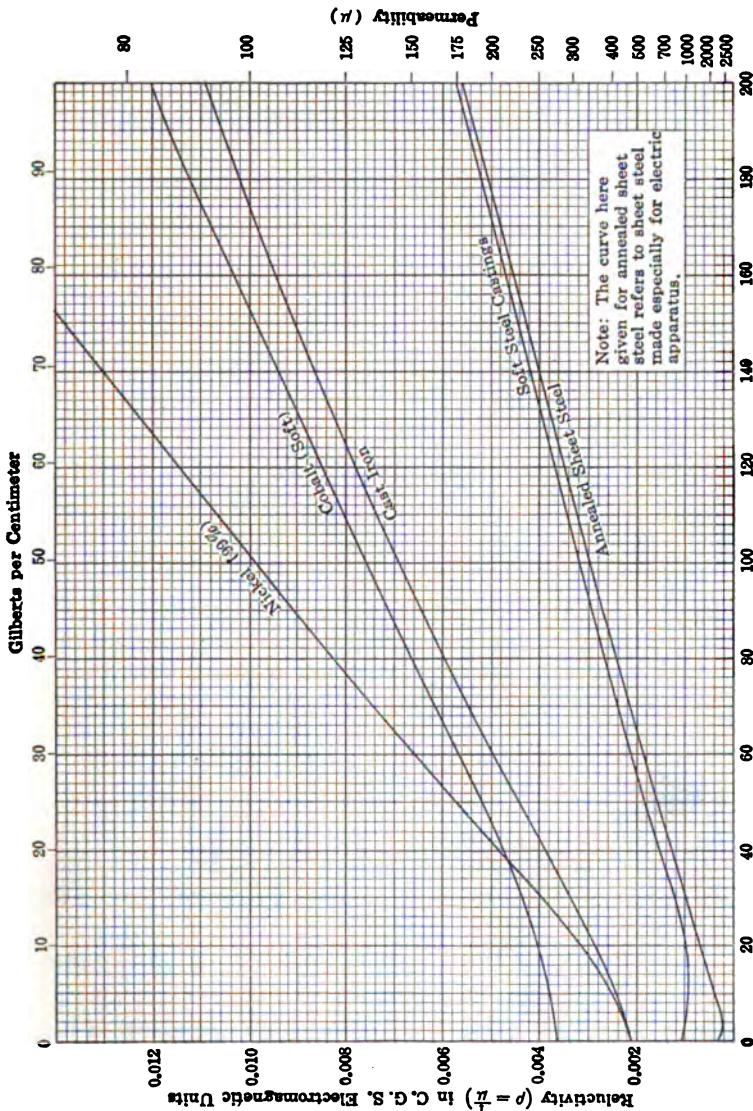


FIG. 77.—Reluctivity and permeability.

calculated from the magnetizing current), will give the locus
of the ends of a set of successively larger hysteresis loops, as

shown by the dotted curve in Fig. 75. Such a curve is sometimes referred to as the "alternating" magnetic characteristic.

Experiment shows that except for the lower values of the flux density, the "rising" and the "alternating" magnetic characteristics practically coincide. For engineering purposes, the alternating magnetic characteristic is the more important. The alternating characteristic may be conveniently referred to as the "normal" *B-H* curve.

Normal *B-H* curves for sheet steel, soft cast steel, cast iron, cobalt and nickel are given in Fig. 76, and the corresponding reluctivity curves $\left(\dfrac{1}{\mu} \text{ against } H\right)$ in Fig. 77. The curves for iron and steel give the average values of the various quantities determined from a large number of tests made in the laboratories of the General Electric Co. The curves for nickel and cobalt are from data given in Steinmetz's "Theory and Calculation of Electric Circuits."

The curves for "annealed sheet steel" refer to low-carbon sheet steel made especially for electrical machinery, and specially annealed. Silicon steel (*i.e.*, steel containing about 3 per cent. of silicon) has a permeability about 10 per cent. less than that of this "standard" sheet steel, except for flux densities below 10,000 gausses. At these lower values of the flux density the permeability of silicon steel may be appreciably greater than that of the "standard" low-carbon sheets. Ordinary sheet iron has a permeability usually much lower than that of "standard" sheet steel.

In general, the shape of the *B-H* curve for a given material is found to depend upon the nature and amount of the impurities in it and the heat treatment to which it has been subjected. Relatively slight differences in the amount of impurity and in the heat treatment may produce a very large variation in magnetic properties. In fact, a wide difference is often found in two samples taken from the same lot of material.

The *B-H* curve also depends upon the temperature of the material, though the variation due to ordinary changes of temperature is slight. For very high temperatures, however, all magnetic substances become practically non-magnetic. This temperature corresponds to the major recalescence point, which is about 750 degrees for "standard" sheet steel.

**113. Magnetic Saturation.**—Experiment shows that up to the highest values of the magnetizing force which have yet been produced, the normal flux density $B$ always increases with an increase in the magnetizing force $H$. However, such experiments indicate that the difference between the flux density $B$ and the magetizing force $H$, that is, the difference $(B\text{-}H)$, approaches a definite maximum limit, dependent solely upon the nature of the material under test. In other words, for any given magnetic substance there is a definite limit to the intensity of magnetization $\left(J = \dfrac{B\text{-}H}{4\pi}\right)$ which can be established in it. When this limiting value of the intensity of magnetization is reached, the substance is said to be "magnetically saturated."

The difference $(B\text{-}H)$, when $B$ and $H$ are both expressed in c.g.s. electromagnetic units, may be conveniently designated as the "metallic" component of the flux density corresponding to a given magnetizing force $H$, since it is equal to the excess of the resultant flux density over the flux density which this same magnetizing force would produce in free space. The maximum, or saturation, value of this metallic flux density is about 20,000 gausses for soft iron and steel, 15,000 gausses for cast iron, 12,000 gausses for cobalt and 6000 gausses for nickel. To increase the flux density beyond these values requires an increase in the magnetizing force equal to the increase in flux density desired.

An inspection of the reluctivity curves in Fig. 77 shows that for values of the magnetizing force about greater than 40 ampereturns per inch (or 20 gilberts per centimeter), the relation between reluctivity and magnetizing force is approximately a straight line one. Due to the fact that there is a definite magnetic saturation point, the reluctivity curves actually have a slight downward droop. However, if instead of the reluctivity $\rho = \dfrac{H}{B}$, the ratio $\dfrac{H}{B-H}$ is plotted against the magnetizing force, it is found that for very high values of $H$, this ratio bears an exact linear relation to the magnetizing force $H$, viz., that

$$\frac{H}{B - H} = \alpha + \sigma H \tag{5}$$

where $\alpha$ and $\sigma$ are constants for any given material. Steinmetz calls the constant $\alpha$ the "coefficient of magnetic hardness," and the constant $\sigma$ the "coefficient of magnetic saturation."

Equation (5) may be written

$$B - H = \frac{H}{\alpha + \sigma H} \qquad (5a)$$

From this relation it is evident that $\frac{1}{\sigma}$ is the limiting value of the metallic component of the flux density, *i.e.*, of $(B - H)$. This follows from the fact that when $H$ becomes very large, $\alpha$ is negligible in comparison with $\sigma H$, and therefore $(B - H)$ becomes equal to $\frac{1}{\sigma}$.

By dividing equation (5a) through by $B$, and putting $\rho = \frac{H}{B} =$ the reluctivity, there results the relation

$$\frac{\rho}{1 - \rho} = \alpha + \sigma H \qquad (6)$$

From Fig. 77, it is seen that the reluctivity of the substances listed is less than 0.02 for all values of $H$ up to 100 gilberts per centimeter (or 200 ampere-turns per inch). Hence for $H$ less than 200 ampere-turns per inch, equation (6) may be written

$$\rho = \alpha + \sigma H \qquad (6a)$$

As is evident from Fig. 77, this straight line relation does not hold for low values of the magnetizing force. It is, however, sufficiently accurate for all practical purposes for all values of $H$ between 20 ampere-turns per inch and 200 ampere-turns per inch, which is the usual range in most electric apparatus and machines.

Problem 1.—(a) Calculate the saturation value of the metallic component of the flux density for annealed sheet steel and cast iron, applying equation (5) to the curves given in Fig. 76 at the points corresponding to $H = 50$ and $H = 90$ gilberts per centimeter respectively. (b) What are the corresponding maximum possible values of the intensity of magnetization?

Answer.—(a) 18,500 gausses for annealed sheet steel and 12,000 gausses for cast iron. (b) 1470 and 956 c.g.s. electromagnetic units respectively.

**114. Magnetic Energy and Hysteresis Loss.**—Experiment shows that energy is always required to establish a magnetic field, and that when a magnetic field disappears, energy is always transferred to some other region or converted into some other form. A magnetic field is, therefore, a source of stored energy, just as a compressed spring is a source of stored energy. The energy stored in a magnetic field is conveniently referred to as "magnetic energy."

A striking manifestation of the energy of a magnetic field is the spark which occurs whenever the flow of electricity in an electric circuit is interrupted, as, for example, by breaking a wire or opening a switch in this circuit. The interruption of the current causes the magnetic field produced by this current to disappear, and the energy thus liberated is transferred to the electric circuit, where it is converted into heat, chiefly in that part of the circuit at which the resistance is the greatest, *i.e.*, in the air gap between the two ends of the circuit. Hence the resulting incandescence of the air at this point. It should be noted, however, that the incandescence of the electric spark is a secondary effect, the primary effect of the potential difference established across the air gap being to produce an actual disruption of the molecules of the air (see Article 148).

As will be shown later (Chapter XI) the mechanical forces between conductors carrying electric currents and between magnets are also manifestations of the energy stored in a magnetic field.

Experiment also shows that when a magnetic flux has once been established, no further transfer of energy to the *magnetic circuit* of this flux is required to maintain the field. This is true even for a magnetic field which is maintained by an electric current. Under such conditions energy is, of course, required to maintain the current, due to the resistance of the *electric* circuit, but none of this energy is dissipated in the *magnetic* circuit. That is, although a steady flow of electricity is always accompanied by a dissipation of heat energy in the path of this flow, an unvarying magnetic flux may exist indefinitely in a magnetic circuit without causing any dissipation of heat energy in this circuit.

However, experiment shows that whenever the flux in the *ferromagnetic* portion of a magnetic circuit is caused to *vary*, either in magnitude or direction, this variation of flux is always accompanied by a *dissipation of heat energy*, irrespective of the nature of this variation, whether it be a variation in magnitude, direction or both. Consequently, the amount of energy *stored* in a magnetic field in a ferromagnetic substance is always less than the total energy required to establish this field, for part of the energy required to establish the field is converted into heat energy, and is not recoverable as useful energy when the field disappears. Similarly, when a magnetic field in a ferromagnetic

substance disappears, only part of the magnetic energy stored in the field is recoverable as useful energy, for part of this stored energy is converted into heat energy upon the disappearance of the field. In both paramagnetic and diamagnetic substances (permeability only slightly different from unity) there is no such dissipation of heat energy, or at least, if there is such, it is inappreciable.

(The dissipation of heat energy referred to in the preceding paragraph is in addition to the heat energy which may be dissipated in the substance in question by any electric current which may be induced in this substance by the varying flux; see Articles 74 and 118.)

The dissipation of heat energy due to the change in the amount or distribution of flux in ferromagnetic substance may be "explained" in terms of a relatively simple hypothesis. This hypothesis is that (1) every molecule of a ferromagnetic substance is itself a permanent magnet, (2) when such a substance is unmagnetized the axes of these molecular magnets have random directions, and, therefore, produce externally no magnetic field, and (3) when a magnetic field is established in such a substance, by some external agent (*e.g.*, an electric current), these molecular magnets turn so that their axes become more or less parallel, just as a magnetic needle tends to take up a direction parallel to the direction of the field in which it is placed. On this hypothesis, any increase in the magnitude of the magnetizing force tends to turn these molecular magnets so that their axes become more nearly parallel, and any change in the direction of the magnetizing force produces a corresponding change in the average direction of their axes. If it is further assumed that (4) the motion of these molecular magnets is opposed by a molecular force of a frictional nature, the production of heat energy when these molecular magnets are caused to change their directions is accounted for. This particular dissipation of heat energy is, therefore, frequently said to be due to "molecular magnetic friction."

As will be shown in Article 117, this particular dissipation of heat energy is intimately related to the phenomenon of magnetic hysteresis, and for this reason the amount of energy so lost is also commonly referred to as the loss of energy due to hysteresis, or briefly, as the "hysteresis loss."

It is of interest to note that the above hypothesis regarding the molecular nature of ferromagnetic substances is also useful as a means of "explaining" a number of other magnetic phenomena. For example, the magnetic saturation of a ferromagnetic substance (see Article 113) may be looked upon as corresponding to the condition when all the molecular magnets have their axes exactly parallel to each other and in the same direction. For any degree of magnetization below this limiting value, the axes of the molecular magnets may be thought of as making acute angles with one another, instead of being actually parallel.

An equivalent hypothesis, which possesses the advantage of coördinating the properties of magnets with those of an electric current, is to assume that in each molecule of a ferromagnetic substance there exists a permanent electric current flowing in a circuit of zero resistance. Such a current would produce a permanent magnetic flux, and would be in every way equivalent to a permanent magnet of molecular dimensions. One or more electrons moving in orbital paths around the center of the molecule (just as the earth moves around the sun) would constitute such a permanent electric current.

However, aside from any hypothesis in regard to the molecular nature of a ferromagnetic substance, the important fact from an engineering standpoint is that a *varying magnetic flux in a ferromagnetic substance is always accompanied by a dissipation of heat energy in this substance*, in addition to the heat energy which may be dissipated within it by other causes, as for example, by electric currents which this varying flux may induce in it.

**115. Total Energy Required to Establish a Magnetic Flux.—** Consider in a magnetic field any volume which is closed on itself and whose walls are tangent at each point to the line of force through that point. Such a volume forms a closed magnetic circuit, in the strict sense of this term (see Article 88). This magnetic circuit may be linked by any number of electric circuits, in some of which the current may be in the right-handed screw direction with respect to the direction of the resultant lines of force, and in others in the left-handed screw direction.

Let $N_1$, $N_2$, etc., be the numbers of turns in these several electric circuits. Let $i_1$, $i_2$, etc., be the currents in these circuits, taken positive when in the right-handed screw direction with

respect to the flux in the magnetic circuit, and negative when in the opposite direction. Let $\varphi$ be the flux in the magnetic circuit, which flux will have the same value at every cross-section of this circuit, for the walls of this circuit are tangent to the lines of force. Let $\mathfrak{R}$ be the reluctance of the magnetic circuit to this flux $\varphi$. Then from the definition of reluctance, when all quantities are expressed in electromagnetic units,

$$\varphi = \frac{4\pi(N_1 i_1 + N_2 i_2 + \text{etc.})}{\mathfrak{R}} \tag{7}$$

Imagine this flux to increase by an amount $d\varphi$ in time $dt$, due to a change in the value of any one or more of the electric currents. Then, from the fundamental law of electromagnetic induction (equation (4) of Article 85), there will be induced in the several electric circuits electromotive forces numerically equal to

$$e_1 = N_1 \frac{d\varphi}{dt}, \quad e_2 = N_2 \frac{d\varphi}{dt}, \qquad \text{etc.} \tag{8}$$

These electromotive forces are each in the left-handed screw direction with respect to the increase in flux (see Article 82), and, therefore, *opposite* in direction to the positive sense of the currents in these circuits. Hence this change in flux corresponds to a conversion of electric energy into some other form (see Article 34), and the amount of electric energy so converted in time $dt$ is[1]

$$dW = e_1 i_1 dt + e_2 i_2 dt + \text{etc.} \tag{9}$$

Substituting in this equation the values of $e_1$, $e_2$, etc., from equation (8) gives

$$dW = (N_1 i_1 + N_2 i_2 + \text{etc.}) \, d\varphi \tag{9a}$$

Substituting in this last equation the value of $(N_1 i_1 + N_2 i_2 + \text{etc.})$ from equation (7) gives

$$dW = \frac{1}{4\pi}\mathfrak{R}\varphi d\varphi \qquad \text{ergs} \tag{9b}$$

When the flux increases by a finite amount, say from the value $\varphi_1$ to the value $\varphi_2$, then, from equation (9b), the total amount

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[1] Note that equation (9) represents an algebraic sum, for in this expression any current which links the flux in the left-handed screw direction is represented by a negative number, and the corresponding product $ei$ is then negative. A negative value of $eidt$ signifies a *gain* of electric energy by this particular circuit. Equation (9), therefore, gives the *net* or *resultant* loss of electric energy from all the electric circuits which link the flux $\varphi$.

of electric energy lost by the currents which link the path of this flux is

$$W = \frac{1}{4\pi} \int_{\varphi_1}^{\varphi_2} \mathfrak{R}\ \varphi d\varphi \qquad \text{ergs} \qquad (9c)$$

The energy represented by this expression is the net, or resultant, loss of electric energy from *all* the electric currents which link the path of the flux $\varphi$, and which is converted into some other form, when this flux increases from the value $\varphi_1$, to the value $\varphi_2$. When the integral is positive, there is an actual conversion of electric energy into some other form; when the integral is negative, there is an actual conversion of some other form of energy into electric energy. When there are two or more electric circuits, one of these may gain and the other lose electric energy, depending upon the directions of the currents with respect to the direction of the flux. However, irrespective of any transfer of electric energy from one circuit to another, a change in the flux which links one or more electric circuits is always accompanied by a conversion of electric energy into some other form, or of some other form of energy into electric energy, and the net amount of energy so converted is equal to the value of $W$ given by equation (9c).

**116. Magnetic Energy in a Circuit of Constant Reluctance.—** When the reluctance $\mathfrak{R}$ of the path of the flux remains constant throughout the change in the flux, equation (9c) may be written

$$W = \frac{1}{8\pi} \mathfrak{R}\ (\varphi_2{}^2 - \varphi_1{}^2) \qquad \text{ergs} \qquad (9d)$$

The conditions which must be fulfilled in order that the reluctance of the path of the flux remain constant are (1) that the permeability of every substance in this path be constant irrespective of the value of the flux density, and (2) that the *relative* distribution of the lines of force which represent this flux remain unaltered (*i.e.*, that the shape of these lines of force do not change during the change in the magnitude or sense of the flux).

The first provision precludes the presence of any ferromagnetic substances in the path of the flux, and the second precludes the relative motion of any current-carrying conductor or magnet (permanent or induced) anywhere in the given magnetic field, and also precludes the presence of magnetic poles anywhere in the field. Under these conditions the change in flux will produce no dissipation of heat energy due to molecular magnetic friction,

since this phenomenon does not occur in bodies of constant permeability. Also, there will be no conversion of electric energy into mechanical energy, for there is no mechanical motion. Hence, the condition that the reluctance $\mathcal{R}$ remain constant throughout the change in flux is equivalent to the condition that this change in flux be accompanied by no production of heat energy or mechanical energy. The conversion of energy is then solely from electric energy to magnetic energy.

Consequently, from equation (9d) the total amount of magnetic energy stored in a volume of *constant reluctance* $\mathcal{R}$, when a flux $\phi$ is established in this volume (*i.e.*, when the flux in this volume is increased from zero to $\phi$) is

$$W = \frac{1}{8\pi}\mathcal{R}\varphi^2 \qquad \text{ergs} \qquad (10)$$

It also follows from equation (9d) that when the flux in a magnetic circuit of *constant reluctance* disappears (*i.e.*, when the flux decreases from $\varphi$ to zero), an equal amount of magnetic energy is liberated. This "liberated" energy must, of course, be converted into some other form of energy; for example, into electric energy in the electric circuit or circuits which link the magnetic circuit.

In the deduction of equation (10) the reluctance $\mathcal{R}$ was taken as the reluctance of the entire closed circuit of the flux $\varphi$. A perfectly rational assumption, fully justified by experiment, is that a flux of given magnitude and given distribution, in a given volume of *constant permeability*, always represents the same amount of *magnetic energy*, irrespective of the nature of the substances outside this volume, and irrespective of how this flux is established. Consequently, equation (10) is applicable not only to a complete magnetic circuit of constant permeability, but also gives the magnetic energy stored in *any* volume in a magnetic field, irrespective of how the flux $\varphi$ is established, *provided* only that the permeability of this particular volume be constant. When this equation is applied to such a portion of a magnetic field, $\mathcal{R}$ is, of course, to be taken as the reluctance of this particular volume and $\varphi$ as the flux through this particular volume.

It is of interest to compare equation (10) for the magnetic energy stored in a reluctance $\mathcal{R}$ with the analogous expression $P = rI^2$ for the power dissipated as heat in a resistance $r$ by a current $I$. Both expressions are of the same mathematical form, but the physical facts represented are entirely different. The

expression $W = \frac{1}{8\pi}\Re\varphi^2$ gives the value of the *energy* which is stored in a reluctance $\Re$ by the magnetic flux $\varphi$, whereas the expression $P = rI^2$ gives the *rate* at which energy is *dissipated* as heat in a resistance $r$ by a current $I$.

When equation (10) is applied to an elementary portion of a magnetic circuit, it may be put in a more convenient form as follows. Let $l$ be the length of this volume parallel to the lines of force, let $S$ be the cross-section of this volume, perpendicular to these lines. The reluctance of such a volume is $\Re = \frac{l}{\mu S}$, and the flux through it is $\varphi = BS$, where $B$ is the flux density in this volume. The substitution of these values of $\Re$ and $\varphi$ in equation (10) gives

$$W = \frac{1}{8\pi}\frac{B^2}{\mu}(lS)$$

But $(lS)$ is the volume of the particular portion of the circuit under consideration. Hence the energy stored in a magnetic field *per unit volume* of this field is

$$w = \frac{B^2}{8\pi\mu} \qquad \text{ergs per cubic centimeter} \qquad (11)$$

where $B$ is the flux density in this volume in gausses and $\mu$ is its permeability.

This relation, like equation (10) from which it is derived, holds only when the permeability of the volume in question is constant, independent of the flux density established in it. This expression is, therefore, not applicable to a volume in a ferromagnetic substance. However, equation (11) is applicable to any part of a magnetic circuit for which the permeability is constant, even though the rest of the circuit is through ferromagnetic substances. For example, this equation gives the energy stored per unit volume of the air gap between any two parts of a magnetic circuit, such as the air gap between the plunger and stop in an electromagnet. Of course, the permeability in such a case is the permeability of the air ($= 1$), not the permeability of the iron.

The energy stored as magnetic energy when a flux is established in a ferromagnetic substance is probably always less than the energy given by equations (10) or (11), due to the phenomenon of molecular magnetic friction (see Article 114).

**Problem 2.**—A certain overhead two-wire transmission line is 100 miles long. The wires are each 0.460 inch in diameter (No. 0000 A.W.G.) and the distance between their centers is 6 feet. The current in the line is 200 amperes.

(a) What is the total magnetomotive force of the loop formed by the two wires? (b) What is the total flux which links this loop (see equation (5b) in Problem 4 of Article 98)? (c) What is the total reluctance of the magnetic circuit of this flux? (d) How much magnetic energy is stored in the magnetic field surrounding the wires? Give answer in watt-seconds (joules) and foot-pounds.

*Answer.*—(a) 251.2 gilberts. (b) 7.72 × 10⁹ maxwells. (c) 0.0325 × 10⁻⁶ oersted. (d) 7720 watt-seconds or 5680 foot-pounds.

**Problem 3.**—The average distance between the pole face and the armature core of a certain 90-kilowatt six-pole generator is ¼ inch. The cross-sec-
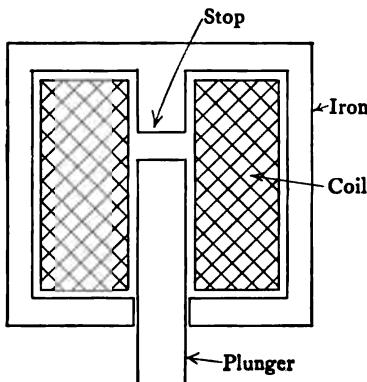


Fig. 78.

tion of each pole face is 64 square inches. The flux per pole is 3.2 × 10⁶ maxwells. Assuming the flux lines in the air gap to be perpendicular to the pole face and uniformly distributed, how many watt-seconds (joules) of magnetic energy are stored in all the six air gaps?

*Answer.*—376 watt-seconds.

**Problem 4.**—Fig. 78 shows a longitudinal section of an "iron-clad" electromagnet with a movable core, or plunger. When the end of the plunger is close to the stop, the lines of force in the air gap between the end of the plunger and the stop may be considered as straight, parallel and uniformly distributed. The cross-sections of the plunger and stop are each 2 square inches, and when the air gap between the two is 0.2 inch, the total flux across this air gap is 200,000 maxwells.

(a) What is the reluctance of the air gap between the plunger and stop? (b) By how much will this reluctance change when the plunger is pulled downward 0.001 inch? (c) By how much will the total energy of the magnetic field of the electromagnet be changed by this motion, provided the

flux remains constant? (*d*) How much mechanical work must be done by the agent which pulls the plunger down this distance? The change in the magnetic energy stored in the plunger is negligible (see Article 117). (*e*) What must be the force exerted on the plunger at this agent? (*f*) What is the *total* upward pull exerted on the plunger due to the current directly and to the magnetic pole induced by this current at the end of the stop? Give answer both in dynes and in pounds. (*g*) Deduce a general formula for the relation between the pull on the plunger in pounds, the flux density in the plunger in kilolines per square inch, and the cross-section of the air gap in square inches, and state the exact conditions under which this formula is applicable. (*h*) If the cross-section of the plunger were reduced to one-half its original value, but the total flux kept constant, by how much would the pull change?

*Answer.*—(*a*) 0.0394 oersted. (*b*) 0.000197 oersted. (*c*) 0.0314 watt-second. (*d*) 0.0314 watt-second or 0.0232 foot-pound. (*e*) 278 pounds. (*f*) 123.7 × 10⁶ dynes or 278 pounds. (*g*)

$$f = 0.0139B^2S \qquad \text{pounds}$$

where $B$ is the flux density in kilolines per square inch and $S$ is the cross-section of the air gap in square inches. This formula holds only when the flux density in the air gap is *uniform* and when the walls of the air gap are perpendicular to the lines of force. (*h*) The pull would be doubled.

**117. Hysteresis Loss and Magnetic Energy in Ferromagnetic Substances.**—As shown in the last article, whenever a magnetic flux $\varphi$ changes from any value $\varphi_1$ to any other value $\varphi_2$, the electric circuits which link its path lose a net amount of electric energy equal to

$$W = \frac{1}{4\pi} \int_{\varphi_1}^{\varphi_2} \Re\varphi\, d\varphi \qquad \text{ergs} \qquad (12)$$

It was also pointed out in that article that, from the principle of the conservation of energy, this loss of electric energy must result in the production of an equal amount of some other form or forms of energy, these other forms of energy being, in the general case, (1) magnetic energy, (2) heat energy due to molecular magnetic friction and (3) mechanical energy or work.

In the last article the above equation was used as a means of deriving an expression for the magnetic energy stored in a magnetic field, when there is no dissipation of heat energy and no change in mechanical energy (*i.e.*, no mechanical work done). In exactly the same manner this same general relation may be used to obtain an expression for the energy dissipated as heat due to molecular magnetic friction, in those special cases in which the change in flux is accompanied by no resultant change in magnetic energy and no change in mechanical energy.

Consider the special case of a magnetic circuit each part of which is at rest with respect to the electric circuits which link it, and in which the lines of force remain fixed in shape throughout any change in flux which may take place. These conditions are satisfied in the case of a closed circular ring of the same material throughout (Fig. 52), when the variation of flux in the ring is caused solely by variations in the intensity (or sense) of an electric current in a uniformly distributed winding completely covering the ring. These conditions are also approximately realized in any core which forms a closed magnetic circuit of uniform permeability, provided this core is at rest with respect to the electric circuits which link it, and provided the leakage of flux from the core into the surrounding air is relatively small.

When there is no motion of any part of the magnetic circuit under consideration, and no motion of any part of the electric circuit, experiment shows that the relation expressed by equation (12) applies not only to the complete (closed) circuit of the magnetic flux $\varphi$, but also to *any portion* of this current. That is, when there is no mechanical work done, equation (12) gives the net amount of energy which is transferred from all the electric circuits in this field to any particular volume in this field, provided $\mathcal{R}$ is taken as the reluctance of this volume and $\varphi$ the flux through it.

Under the specific conditions just stated, viz., no mechanical motion and no change in the shape of the lines of force, equation (12) may be put into a more convenient form, as follows: Consider in such a magnetic circuit an elementary volume of length $l$ parallel to the line of force through it, and of cross-section $S$ perpendicular to this line of force. The product $\mathcal{R}\varphi$ for such a volume, *i.e.*, the reluctance drop through this volume, is equal to $Hl$ (see equation (19) of Article 90). The flux through this volume is $BS$, and, therefore, $d\varphi = SdB$. Whence, for this particular volume

$$\mathcal{R}\varphi d\varphi = (Hl)(SdB) = (lS)HdB$$

Hence the product $\mathcal{R}\varphi d\varphi$ *per unit volume* of the field is

$$\frac{\mathcal{R}\varphi d\varphi}{lS} = HdB$$

Therefore, the total magnetic energy and heat energy developed per unit volume of a magnetic field, when the change in flux is accompanied by no mechanical motion and no change in the shape of the lines of force, is

$$w = \frac{1}{4\pi} \int_{B_1}^{B_1} H dB \qquad \text{ergs per cubic centimeter} \qquad (13)$$

where $B_1$ and $B_2$ are respectively the initial and final values of the flux density in this particular volume. From this relation it is seen that for a given change in flux density $B$, the change in energy depends not only upon the initial and final values of $B$, but also upon the relation between the flux density $B$ and magnetizing force $H$ during this change.

The relation between the flux density $B$ and the magnetizing force $H$ for any given change in the magnetic state of a given sample of material may always be determined by the step-by-step
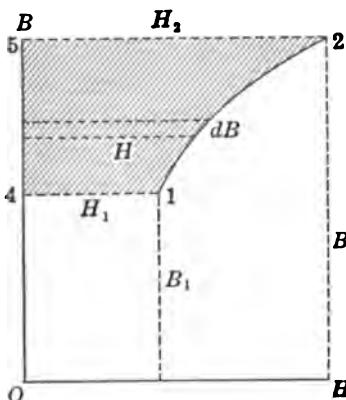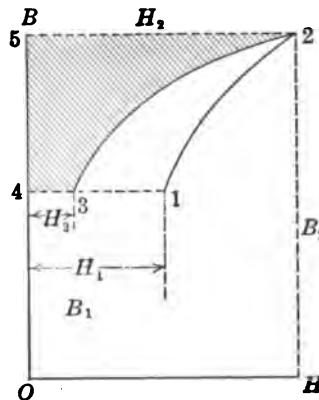


FIG. 79A.                    FIG. 79B.

method explained in Article 110. Plot the values of $B$ and $H$ as thus determined, taking values of $B$ as ordinates and values of $H$ as abscissas, as indicated in Fig. 79A. Then the value of the integral $\int_{B_1}^{B_2} H dB$, for the particular change in state represented by the curve thus found, is equal to the area on this $B$-$H$ diagram included between the two abscissas $H_1$ and $H_2$, the given curve and the $B$-axis, namely, the shaded area 1-2-5-4 in Fig. 79A. From equation (13), this area multiplied by $\frac{1}{4}\pi$ is then equal to the sum of (a) the magnetic energy transferred to the given unit volume plus (b) the heat energy dissipated in this unit volume due to molecular magnetic friction, for this particular change in the magnetic condition of the sample under test.

In the case of all ferromagnetic substances, as pointed out in Article 111, the flux density always decreases less rapidly for a

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decreasing magnetizing force than it increases for an increasing magnetizing force. Consequently, to reduce the flux density from the value $B_2$ to its original value $B_1$, the magnetizing force must in general be reduced to a value $H_3$ smaller than its original value $H_1$, and the relation between $B$ and $H$ during this reverse change will be represented by some such curve as that connecting the points 2 and 3 in Fig. 79$B$. For this change in the flux density from $B_2$ to $B_1$, the product $H dB$ will be negative, since $dB$ here represents an actual decrease, or a "negative increase." The area 2-3-4-5 in Fig. 79$B$, multiplied by $\frac{1}{4}\pi$, therefore, represents the energy transferred *from* the given unit volume of the magnetic circuit to the electric circuit or circuits which link this magnetic circuit.

The area 2-3-4-5, however, does not represent the total amount of magnetic energy lost by this unit volume during the decrease in flux density from $B_2$ to $B_1$, for part of the magnetic energy lost during the decrease in flux density from $B_2$ to $B_1$, is converted into heat energy in the particular unit volume in question, due to molecular magnetic friction. Just how much *magnetic* energy is gained or lost by a unit volume of a ferromagnetic substance, for a given change in flux density, cannot in general be determined from the relations thus far stated. The only general conclusion that may be drawn is that $\frac{1}{4}\pi$ times the area 1-2-5-4 is equal to the magnetic energy $w_m$ gained by the unit volume, *plus* the heat energy $w_h$ developed in it, during the increase in the flux density from $B_1$ to $B_2$, and that $\frac{1}{4}\pi$ times the area 2-3-4-5 is equal to the magnetic energy $w'_m$ lost by the unit volume *less* the heat energy $w'_h$ developed in it during the decrease in flux density from $B_2$ to $B_1$, viz.,

$$\frac{1}{4\pi} \text{ (Area 1-2-5-4)} = w_m + w_h$$

$$\frac{1}{4\pi} \text{ (Area 2-3-4-5)} = w'_m - w'_h$$

Therefore, $\frac{1}{4}\pi$ times the difference in these two areas, namely, $\frac{1}{4}\pi$ times the area 1-2-3, is equal to the difference between the gain and the loss of magnetic energy, plus the total amount of heat energy developed, during the given increase and decrease of the flux density, viz.,

$$\frac{1}{4\pi} \text{ (Area 1-2-3)} = w_m - w'_m + (w_h + w'_h) \tag{14}$$

As shown in Fig. 79*A*, the increase in the magnetizing force required to increase the flux density from any value $B_1$ to some other value $B_2$ is in general less than the amount by which the magnetizing force must be decreased in order to produce the reverse change in the flux density. However, experiment shows that after a number of repetitions of the same cycle of changes in flux density (say from $B_1$ to $B_2$ and then back again from $B_2$ to $B_1$, and so on repeatedly) the maximum value of the flux density always corresponds to the *same* maximum value of the magnetizing force, and the minimum value of the flux density always corresponds to the *same* minimum value of the magnetizing force. In other words, the ascending *B-H* curve and the descending *B-H* curve coincide at their ends, and form a *closed* loop. As already noted, such a loop is called a "hysteresis loop." A symmetrical hysteresis loop, *i.e.*, one whose ends correspond to numerically equal but opposite values of the flux density, is shown in Fig. 74.

Experiment justifies the assumption that the magnetic state corresponding to any given point on a closed hysteresis loop is characterized by a definite amount of *magnetic* energy. That is, when the magnetizing force and the flux density *each* passes through a complete cycle of values, the magnetic energy at the end of the given cycle is equal to the magnetic energy at the beginning of this cycle. Consequently, from equation (14) it follows that $\frac{1}{4\pi}$ times the area of a *closed* hysteresis loop is equal to the total *heat* energy developed per unit volume, due to molecular magnetic friction, during the corresponding cycle of changes in the flux density and magnetizing force. As already noted, the energy thus dissipated as heat is called the "hysteresis loss."

Hence *the hysteresis loss per cycle per cubic centimeter* in any magnetic substance, due to a complete cycle of changes in the flux density and the magnetizing force, *is equal to $\frac{1}{4\pi}$ times the area of the corresponding hysteresis loop*, provided this loop is determined under such conditions that there is no mechanical motion and no change in the *relative* distribution of the lines of force.

The relation just stated is on the assumption that unit distance on the ordinate scale of the hysteresis loop represents a flux density of 1 gauss, and unit distance on the abscissas scale represents 1 gilbert per centimeter. When unit distance on the

ordinate scale is taken equal to $b$ gausses, and unit distance on the abscissa scale equal to $h$ gilberts per centimeter, and the actual area of the hysteresis loop is $A$, then

$$\text{Hysteresis loss} = \frac{bh}{4\pi}A \qquad \text{ergs per cu. cm. per cycle} \qquad (15)$$

The hysteresis loss in iron and other magnetic materials is an important factor in the design of all apparatus and machines whose operation depends upon a varying magnetic flux. In fact, the hysteresis loss in a ferromagnetic substance is usually of more importance, from a practical standpoint, than its permeability. The variation of the hysteresis loss with the limiting values of the flux density and with the quality of the iron or steel employed has, therefore, been extensively investigated.

The results of these investigations may be summarized in a relatively simple empirical law, first stated by Steinmetz, viz., that in any given volume of iron or steel the hysteresis loss per unit volume per cycle varies as the 1.6th power of the maximum flux density, provided the cycle corresponds to a symmetrical hysteresis loop. That is, calling $B_m$ the maximum value of the flux density in gausses during the given cycle (*i.e.*, the numerical value of flux density corresponding to either of the two pointed ends of the hysteresis loop), the hysteresis loss is

$$w_h = \eta B_m^{1.6} \qquad \text{ergs per cu. cm. per cycle} \qquad (16)$$

The coefficient $\eta$, which depends on the nature of the substance in question, is called by Steinmetz the "coefficient of hysteresis" of this particular substance. For annealed sheet steel the value of the hysteresis coefficient ranges from about 0.001 to 0.002. For annealed sheet steel containing about 3 per cent. silicon its value is about 30 per cent. less.

It should be carefully noted that equation (16) is not an exact law (like Ohm's Law, for example), but is only an approximate representation of the actual facts. However, for flux densities between about 1000 and 12,000 gausses, this relation is sufficiently accurate for most practical purposes. At higher flux densities the hysteresis loss increases faster than the 1.6th power of the flux density, particularly if the material under test is non-homogeneous. It should also be noted that the equation (16) is applicable only to variations in flux density between equal and opposite maximum values, *i.e.*, to symmetrical hysteresis loops, and then only when the change from $-B_m$ to $+B_m$ is a continuous increase,

and the reverse change a continuous decrease in the flux density[1] (see Chapter IV of Steinmetz's "Theory and Calculation of Electric Circuits").

The hysteresis coefficient $\eta$, like the permeability, depends not only upon the chemical composition but also upon the physical characteristics of the substance in question. In general, hard steel has a much higher hysteresis coefficient than soft steel. The sheet steel used for armature cores, and for other parts of magnetic circuits carrying a varying flux, is, therefore, always carefully annealed. The reduction in hysteresis loss by annealing may be 50 per cent. or more.

When iron or steel is exposed continuously to temperatures above 100°C. the hysteresis coefficient, and, therefore, the hysteresis loss, may undergo a gradual increase. This phenomenon is known as magnetic "ageing." Silicon steel is practically free from ageing. The ageing of other steels depends upon their composition and heat treatment. The ageing of sheet steel made especially for electric apparatus, when of proper chemical composition and properly annealed, is practically negligible. Slight variations in chemical composition or heat treatment, however, may produce appreciable ageing.

As already noted, a dissipation of energy as heat due to molecular magnetic friction takes place not only when the flux density in a ferromagnetic substance changes in magnitude and sense, but also whenever there is a change in the direction of the lines of force in such a substance with respect to a line fixed in this body, as, for example, when a piece of iron is rotated in a stationary magnetic field. Through a surface of unit area fixed in a body which rotates in a magnetic field of constant flux density $B$, the flux varies continuously from this value $B$ to an equal value in the opposite direction and back again, repeating this cycle for every revolution. Experiment shows that under these conditions the energy dissipated per unit volume as heat, due to molecular magnetic friction, is exactly the same as would be dissipated were the magnetic substance kept at rest and the flux density varied between these same limits by varying its magnitude and sense.

The energy which is dissipated as heat in this case supplied

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[1] If during the change from one value to another there is a subsidiary loop like the dotted loop shown in Fig. 74, the hysteresis loss is increased by an amount proportional to the area of this subsidiary loop.

as mechanical work by the agent which produces this motion. When the flux density of the field and the hysteresis coefficient of the substance in question are known, the mechanical work required to produce this "loss" of energy may be calculated from the Steinmetz formula.

It should be noted that equation (13), when applied to a substance of *constant* permeability $\mu$ (and, therefore, free from molecular magnetic friction), gives for the total energy developed per unit volume, when a flux density $B$ is established in this volume the value

$$w = \frac{1}{4\pi}\int_0^B HdB = \frac{1}{4\pi\mu}\int_0^B BdB = \frac{B^2}{8\pi\mu}$$

This is identical with equation (11), as it should be, since when the permeability is constant, *all* the energy transferred to the field is stored as *magnetic* energy.

In the case of ferromagnetic substances, however, equation (13) gives the change in the magnetic energy per unit volume *plus* the heat energy developed, when the flux density changes from $B_1$ to $B_2$. In particular, when the flux density changes from zero to any value $B$, the integral

$$w = \frac{1}{4\pi}\int^B HdB$$

is equal to the magnetic energy stored per unit volume during this change plus the heat energy developed in this volume during this change. Even when the substance is originally unmagnetized, and the change in magnetic state corresponds to the "rising" *B-H* characteristic (see Fig. 80), it is probable that some heat energy is developed in magnetizing the body. Just how much heat is developed under such conditions is a matter of experiment, but, as far as the author is aware, this has never been determined.

From equation (13), the *total* energy input per unit volume of the given magnetic substance, when it is magnetized to a flux density $B$, starting from an originally unmagnetized state, is equal to $\frac{1}{4\pi}$ times the shaded area in Fig. 80. This area, when the flux density is well up on the *B-H* curve, is always less than the area of the triangle 0-1-2. The area of this triangle is $\frac{1}{2}BH$, where $H$ is the magnetizing force corresponding to the flux density $B$. Or, since $B = \mu H$, where $\mu$ is the permeability corresponding to the flux density $B$, the area of this triangle is $\frac{1}{2}\frac{B^2}{\mu}$.

Hence the *total* energy input per unit volume, when the flux density is well up on the *B-H* curve, is always less than $\dfrac{B^2}{8\pi\mu}$ ergs per cubic centimeter. Consequently the *magnetic* energy stored per unit volume must also be less than $\dfrac{B^2}{8\pi\mu}$, since this magnetic energy cannot be greater than the total energy. It is also probable that even for the lower values of the flux density, the magnetic energy per unit volume is less than $\dfrac{B^2}{8\pi\mu}$.

One important conclusion which may be drawn from these considerations is that, for a given flux density $B$, the magnetic energy
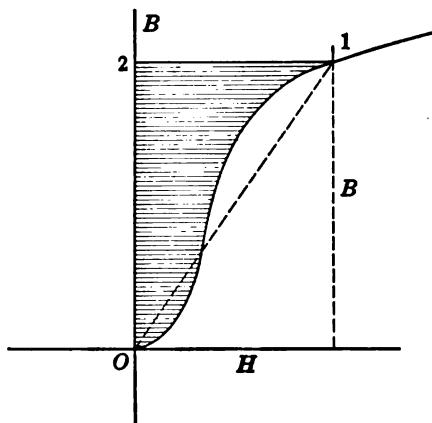


FIG. 80.

per unit volume of a substance whose permeability is greater than unity is *always less* than the magnetic energy per unit volume of a non-magnetic substance. In fact, in any magnetic circuit which is partly iron and partly air, even though the length of the air gap is only a very small part of the total length of the circuit, practically all the magnetic energy is stored in the air gap, and only a relatively small amount of energy is stored in the iron. In other words, where the reluctance is greatest the magnetic energy is greatest (compare with equation (10)).

Problem 5.—(*a*) What is the hysteresis loss per cubic centimeter per cycle corresponding to the hysteresis loop shown in Fig. 74? (*b*) What is the hysteresis coefficient of this sample?

*Answer.*—(*a*) 11,500 ergs. (*b*) 0.00239.

**Problem 6.**—In a certain laminated steel core the flux is caused to alternate continuously between equal and opposite values, making 60 complete cycles (or 120 changes in direction) per second. The cross-section of this core is 30 square inches and its length is 15 inches. The maximum value of the flux density during each cycle of variation is 50 kilolines per square inch. The hysteresis coefficient of the steel is 0.0012.

What is the power (energy per second) dissipated in this core due to hysteresis?

*Answer.*—89 watts.

**Problem 7.**—The flux density established at each point in the armature core of a certain motor by an unvarying (direct) current in the field winding is 75 kilolines per square inch. The volume of this core is 2 cubic feet. The hysteresis coefficient of the sheet steel of which the core is made is 0.001. The core has no winding on it. By means of a crank attached to the armature shaft a force can be applied by hand to turn the core. The lever arm of this crank is 1 foot.

Neglecting all mechanical friction, what is the average force in pounds which must be applied to this crank in order to turn the armature at a constant speed?

*Answer.*—2.13 pounds.

(NOTE.—When the flux density varies appreciably from point to point in the core, as would ordinarily be the case in practice, the total hysteresis loss cannot be calculated by taking for $B_m$ the average value of the maximum flux density, and multiplying the loss per unit volume by the total volume of the core. Instead, the *average value of the* 1.6th *power* of the maximum flux density would have to be used as the value of $B_m^{1.6}$ in equation (16) in order to obtain the *average* loss per unit volume.)

## 118. Eddy Currents and Eddy-current Loss.—A varying magnetic flux induces an electromotive force in every path which links this flux; this is the fundamental law of electromagnetic induction (see Article 81). *Every* conducting path which links a varying magnetic flux, and which contains no other source of electromotive force, must have a current induced in it equal at each instant to the electromotive force induced by this varying flux, divided by the resistance of the path.

The iron and steel portions of the magnetic circuits of electric apparatus and machines are conductors. Consequently, whenever the lines of force in an iron (or other ferromagnetic) core vary with time, either in number or in direction with respect to any fixed line in this core, electric currents are induced in this core. These currents result in a dissipation of energy in the core, for a current $i$ in a resistance $r$ always causes a dissipation of energy as heat at a rate equal to $ri^2$. The loss of energy due

to these induced currents in the magnetic circuit is in addition to the hysteresis loss.

The electric currents which are induced in any conducting mass due solely to the line of force *which pass through the space occupied by this mass*, as distinguished from the current which may be induced in this conductor by lines of force which *link it*, but which do not pass through the space occupied by it, are called "eddy currents," or "Foucault currents." The conducting mass may form no part of the main electric circuit, as is the case with the iron and steel parts of the magnetic circuits of apparatus and machines. In this case the only currents in the conducting mass are the eddy currents. Eddy currents, however, may also be induced in the copper or other conductors which form the main electric circuits. In this case, the *resultant* current in the given conductor is the sum of the main current and the eddy currents.

The power which is dissipated in a conductor as heat, due to the existence of eddy currents in it, is called the "eddy-current loss." When there are no other currents in the conductor, as is the case in the iron and steel parts of magnetic circuits, the eddy-current loss, in the path of an eddy current of intensity $i$, is $ri^2$, where $r$ is the resistance of the path. When the conducting mass in which the eddy currents are induced is also the path of some other current, the eddy currents may merely alter the *distribution* of the current over the cross-section of the conductor, without changing the magnitude of the total current. This alternation in the distribution of the current, however, always increases the heat energy developed, thereby making the "effective" resistance of the conductor to the main current greater than its ohmic, or d.c., resistance (see Article 178).

It is to reduce the amount of heat energy developed by eddy currents that the iron or steel parts of a magnetic circuit are "laminated," *i.e.*, built up of thin sheets. In order to secure the maximum reduction of the eddy-current loss, the laminations which make up the core are always so arranged that their planes are practically parallel to the lines of force through them; *i.e.*, the lines of force enter the laminations "edgewise." It should be particularly noted that the lamination of a core has no effect whatever on the hysteresis loss;[1] the core is laminated solely for the purpose of reducing the eddy-current loss.

[1] This, of course, on the assumption that each lamination has the same chemical and physical characteristics as a solid core would have.

Of course, it is necessary to laminate only those parts of a magnetic circuit in which eddy currents are induced, *i.e.*, those parts in which the flux varies in magnitude or direction with respect to the core. For example, the field cores and yokes of a direct-current dynamo need not be laminated, but the armature core must be, since the flux through each elementary volume of the core reverses in direction, with respect to the core, as the core rotates in the field. Due to the pulsation in the flux caused by the teeth of the armature core, as they rotate under the faces of the field poles, eddy currents are also induced in these pole
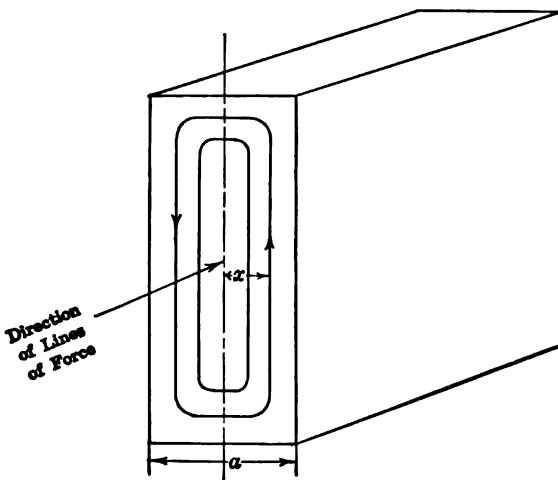


Fig. 81.

faces. The pole faces are, therefore, also frequently laminated. The thickness of sheet steel used for laminations ranges from 0.014 to 0.025 inch.

An approximate expression for the power dissipated as heat in a laminated core, due to eddy currents, may be deduced as follows. Consider first the case of a stationary core in which the lines of force are parallel and remain fixed in direction and in which the variation in flux is due solely to a variation in the magnitude and sense of the flux density. This condition is realized when a straight core is magnetized by an alternating current in a winding placed around it.

When the laminations which make up the core are thoroughly insulated from one another, the eddy-current stream lines will

form closed loops confined to individual laminations, *i.e.*, there will be no flow of electricity from one lamination to the next. The shape of the eddy-current stream lines in a single lamination is indicated in Fig. 81. The planes of the loops formed by these stream lines are perpendicular to the lines of force, and the long sides of these loops are parallel to the surface of the lamination, as indicated.

Consider the eddy-current stream line whose long side is at a distance of $x$ centimeters from the central plane of the lamination. Neglecting the curvature at the two ends of the loop formed by this stream line, this loop may be considered as a rectangle having a width of $2x$ centimeters and a length of $y$ centimeters. When the flux density $B$ has the same value at each point in this rectangle, the total flux through it is $\varphi = 2xyB$ maxwells. Hence, when the flux density is changing at the rate of $dB$ gausses in $dt$ seconds, the electromotive force induced in the particular stream line which forms this rectangle is

$$e = \frac{d\varphi}{dt} = 2xy \frac{dB}{dt} \qquad \text{abvolts}$$

This electromotive force induces in the loop formed by this rectangle a current whose density must be such that the total resistance drop around the closed path formed by this stream line is equal to this electromotive force. Let $\sigma$ be the density of this induced current, in abamperes per square centimeter, at any point in the long side of the rectangle formed by the stream line, and let $\rho$ be the resistivity of the lamination in abohm-centimeters. Then from equation (15) of Article 64, the resistance drop per unit length of each long side is $\rho\sigma$. The total resistance drop in both of the long sides is, therefore, $2\rho\sigma y$ abvolts. When the rectangle is very thin, as is here assumed, the resistance drop in the two ends of the rectangular loop may be neglected, and $2\rho\sigma y$ may then be taken as the total drop around this rectangle. Hence, equating $2\rho\sigma y$ to the value of $e$ just found, the current density must have the value

$$\sigma = \frac{x}{\rho} \frac{dB}{dt} \qquad \text{abamperes per sq. cm.} \qquad (17)$$

That is, except for points near the edge, the current density at any point in the lamination is (*a*) directly proportional to the distance of this point from the central plane of the lamination, (*b*) inversely proportional to the resistivity of the lamination,

and (c) directly proportional to the time-rate of change of the flux density. Note particularly that at the center of the lamination there is no eddy current, and that at the surface of the lamination the density of the eddy current is a maximum.

The resistance of an elementary cube of any conductor to a current which is perpendicular to any one of its faces is $\rho \dfrac{dx}{(dx)^2}$ $= \dfrac{\rho}{dx}$, where $dx$ is the length of the edge of this cube and $\rho$ is its resistivity. The current through such a cube is $\sigma(dx)^2$ where $\sigma$ is the current density. Hence the rate at which heat is dissipated in an elementary cube of volume $(dx)^3$, by a current whose density is $\sigma$, is

$$\frac{\rho}{dx}\left[\sigma(dx)^2\right]^2 = \rho^2\sigma\,(dx)^3$$

Whence, the rate at which heat energy is developed *per unit volume* at any point in a conductor is equal to $\rho\sigma^2$, where $\rho$ is the resistivity at this point and $\sigma$ the current density at this point.

Applying this general relation to the lamination under consideration, the rate at which heat is developed per cubic centimeter at any point[1] which is at a distance $x$ from the central plane of the lamination is

$$\rho\sigma^2 = \frac{x^2}{\rho}\left(\frac{dB}{dt}\right)^2 \qquad \text{ergs per cu. cm. per second.}$$

At the center of the lamination the heat developed is, therefore, zero and at the surface a maximum.

The *average* rate at which heat is developed per cubic centimeter for the entire cross-section of the lamination is the average of this expression between the limits $x = 0$ and $x = \dfrac{a}{2}$. On the assumption that the flux density is constant over the cross-section, $\dfrac{dB}{dt}$ likewise has the same value at every point of this cross-section, and, therefore, the average of $\rho\sigma^2$ over this cross-section is equal to $\dfrac{1}{\rho}\left(\dfrac{dB^2}{dt}\right)$ times the average of $x^2$ between the limits $x = 0$ and $x = \dfrac{a}{2}$. But the average of $x^2$ between the limits $x = 0$ and $x = \dfrac{a}{2}$ is equal to the integral of $x^2$ between these

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[1] Except near the edges of the lamination.

two limits divided by $\frac{a}{2}$. The integral of $x^2$ between these limits

is $\frac{1}{3}\left[\left(\frac{a}{2}\right)^3 - (0)^3\right] = \frac{a^3}{24}$, and this divided by $\frac{a}{2}$ gives $\frac{a^2}{12}$.

Hence, the average rate at which heat energy is dissipated per unit volume of the lamination, *at any given instant of time*, due to the eddy currents in it, is

$$p_e = \frac{a^2}{12\rho}\left(\frac{dB}{dt}\right)^2 \qquad \text{ergs per cu. cm. per second} \qquad (18)$$

This expression not strictly correct, since no account is taken of the fact that for points near the edge of the plate the current density is not equal to the value given by equation (17). However, for any lamination whose width is, say, 20 times its thickness, the error involved is practically negligible.

From equation (18) it is seen that the eddy-current loss varies (a) directly as the square of the thickness of the laminations, (b) inversely as their resistivity, and (c) directly as the square of the *rate of change* of the flux density. Hence the *thinner* the laminations and the *higher* their resistivity the less is the eddy-current loss. Also, the more rapidly the flux density varies with time, the greater is the eddy-current loss.

Equation (18) gives the power loss *at any instant* during the variation in the flux density. For a complete cycle of variation, *e.g.*, from a maximum value $B_m$ in one direction to an equal value in the opposite direction and back again, the *average* rate at which the eddy currents develop heat is equal to $\frac{a^2}{12}$ times the average value of the square of $\frac{dB}{dt}$ for this cycle. The average value of $\frac{dB}{dt}$ will, of course, depend upon the manner in which the flux density varies with time during this cycle.

Consider the special case, which is usually more or less closely approximated in practice, when the variation of the flux density with time may be represented by the sine of an angle which increases uniformly with time, viz., when

$$B = B_m \sin\left(\frac{2\pi t}{T}\right) \qquad (19)$$

where $B_m$ is the maximum value of the flux density during the cycle, and $T$ is the time required for the flux density to pass

through a complete cycle of values. This time interval $T$ is called the "period" of the given cycle. The reciprocal of $T$, namely, the number of times the given cycle of values is passed through in unit time, is called the "frequency" of the cycle. In terms of the frequency, which is usually represented by the symbol $f$, equation (19) may also be written

$$B = B_m \sin (2\pi ft) \qquad (19a)$$

Any quantity which varies with time in the manner just described may be conveniently described as varying "sinusoidally" with time.

The student should note carefully that in both equation (19) and in equation (19a) the angle is in *radians*, not in degrees. For example, when the frequency is 25 cycles per second, the value of the flux density $\frac{1}{200}$ second after it passes through its zero value is $B_m$ times the sine of $2\pi \times 25 \times \dfrac{1}{200} = 0.785$ radians.

Since 1 radian $= \dfrac{360}{2\pi} = 57.3$ degrees, this angle of 0.785 radians is equal to $0.785 \times 57.3 = 45$ degrees.

When the flux density in the core varies sinusoidally with time, at a frequency of $f$ cycles per second, the time rate of change of the flux density at any instant is

$$\frac{dB}{dt} = \frac{d}{dt}\left[ B_m \sin (2\pi ft) \right] = 2\pi f B_m \cos (2\pi ft) \qquad (20)$$

where $B_m$ is the maximum value of the flux density during the cycle. The average value of the square of $\dfrac{dB}{dt}$ for a complete cycle is then $(2\pi f B_m)^2$ times the average value of the square of the cosine of $(2\pi ft)$ during this cycle, *i.e.*, during the time required for $t$ to change from zero to $T = \dfrac{1}{f}$. When $t$ changes from zero to the value $\dfrac{1}{f}$, the angle $(2\pi ft)$ changes from zero to $2\pi$. Hence the average value of the square of the cosine of $(2\pi ft)$ between the limits $t = 0$ and $t = \dfrac{1}{f}$ is the same as the average value of $\cos^2 x$, between the limits $x = 0$ and $x = 2\pi$, where $x$ is any angle which increases continuously from zero to $2\pi$.

The average value of $\cos^2 x$ between the limits $x = 0$ and $x = 2\pi$ is equal to $\frac{1}{2}$. This may be proved as follows. Note first

that $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\cos^2 x$, and, therefore, $\cos^2 x = \frac{1}{2}(1 - \cos 2x)$. Hence the average value of $\cos^2 x$ is equal to $\frac{1}{2}$ times the average value of $(1 - \cos 2x)$. The average value of $\cos 2x$ between the limits $x = 0$ and $x = 2\pi$ is zero, since for every positive value of the cosine there is within these limits an equal negative value. Hence the average value of $\cos^2 x$ is equal to $\frac{1}{2}$.

Consequently, when the flux density in a laminated core varies sinusoidally with time between equal and opposite maximum values, the average amount of power dissipated as heat by the eddy currents induced in it, is

$$P_e = \frac{\pi^2}{6\rho}\,(afB_m)^2 \qquad \text{ergs per cu. cm. per second} \qquad (21)$$

provided (a) that the lines of force are parallel to the planes of the laminations, (b) that the laminations are thoroughly insulated from each other, and (c) that the thickness of each lamination is small in comparison with its other dimensions.

The three conditions just stated are seldom fully realized in practice. In particular, the laminations are seldom thoroughly insulated from each other. Frequently the only insulation between laminations is the natural scale which forms on the sheets when they are annealed. This scale forms a surprisingly good insulation, but does not entirely prevent a flow of electricity from one lamination to the next. Even when the laminations are varnished before being built into a core, the insulation between adjacent laminations is not perfect at every point. Consequently, equation (21) usually gives too low an eddy-current loss. This is usually taken into account in practice by writing this formula

$$P_e = \epsilon(afB_m)^2 \text{ ergs per cu. cm. per second} \qquad (21a)$$

and determining the coefficient $\epsilon$ experimentally. This coefficient $\epsilon$ is called the "eddy-current coefficient;" compare with the hysteresis coefficient $\eta$ (Article 117).

Comparing equations (21) and (21a) it is seen that the eddy-current coefficient $\epsilon$ varies inversely as the resistivity. For example, steel which contains about 3 per cent. of silicon has a resistivity about three times that of ordinary "standard" electrical sheet steel. Hence the eddy-current coefficient for silicon steel is about one-third of the value of this coefficient for "standard" sheet steel, and the eddy-current loss is correspondingly less. This is the chief advantage of silicon steel over "standard" sheet steel. Also note that since the resistivity of

steel increases with increase of temperature, the eddy-current loss *decreases* with increase in temperature.

Average values of the eddy-current coefficient, for a sinusoidal variation of the flux density, and normal working temperature (about 75°C.), are as follows:

> For standard sheet steel (annealed)      0.00022
> For silicon sheet steel (annealed)       0.000065

These values apply only when all quantities in equation (21a) are expressed in c.g.s. electromagnetic units. Variations of 25 per cent. or more from these average values may occur, depending upon the physical and chemical characteristics of the material in question. The eddy-current coefficient also depends on the manner in which the flux density varies with time, how thoroughly the laminations are insulated from one another, and upon how tightly they are pressed together.

Equations (21) and (21a) are deduced above on the assumption that (a) the core is stationary, (b) that the lines of force have a fixed direction with respect to this core, and (c) that the change in the flux which threads any area in the core is due to variations in the magnitude and sense of the flux density, and not to changes in the direction of the lines of force. Eddy currents, however, are also in general induced in a conductor whenever it moves in a magnetic field, or whenever the lines of force through this conductor change in direction, even though the flux density may remain constant in magnitude.

For example, eddy currents are always induced in the armature of a dynamo, irrespective of whether the armature or the field structure is the moving member. That this is true may readily be seen by imagining in the core a closed path fixed with respect to the core. When the core moves with respect to the field, or when the field moves with respect to the core, the lines of force will link this closed circuit first in one direction, and then in the opposite direction, and, therefore, induce an alternating current in this circuit.

It may be shown, by an analysis similar to that given above, that the eddy-current loss in a laminated core which rotates in a magnetic field of constant flux density, or in which a magnetic field of constant flux density continually changes in direction, is also given by equation (21a), when for $B_m$ is taken the value of

this constant flux density and for $f$ is taken the number of times per second that the flux which links a circuit *fixed in the core* passes through a complete cycle of values.

**Problem 8.**—The resistivity of a certain sample of sheet steel is 10 microhm-centimeters at 0°C. Its resistivity temperature coefficient is 0.006 per degree Centigrade. A laminated core is built up of this steel, and the laminations are thoroughly insulated from one another.

(a) What is the eddy-current coefficient of this core at 20°C.? (b) At 90°C.?

*Answer.*—(a) 0.000147. (b) 0.000107.

**Problem 9.**—The laminations in the core described in Problem 8 are 0.0188 inch thick. The total volume of this core is 300 cubic inches. A magnetic field which varies sinusoidally with time, at a frequency of 60 cycles per second, is established in this core. The maximum flux density during each cycle is 70 kilolines per square inch, and has the same value at each point in the core.

(a) What is the total eddy-current loss in watts, when the temperature of the core is 20°C.? (b) When the temperature of the core is 90°C.?

*Answers.*—(a) 70.2 watts. (b) 51.1 watts.

**Problem 10.**—A circular rod which has a radius of $r$ centimeters and a length of $l$ centimeters is placed with its axis parallel to the lines of force in a uniform magnetic field. The resistivity of the rod is $\rho$ abohm-centimeters.

(a) Prove that when the flux density at each point of this field is changing at the rate of $\dfrac{dB}{dt}$ gausses per second, the density of the eddy current induced in this rod, at any point at a distance $x$ from its axis, is

$$\sigma = \frac{x}{2\rho}\frac{dB}{dt} \qquad \text{abamperes per square centimeter.}$$

(b) Prove that the total power dissipated in the entire rod at any instant is

$$p_s = \frac{\pi l r^4}{8\rho}\left(\frac{dB}{dt}\right)^2 \qquad \text{ergs per second.}$$

(c) Prove that when the flux density varies sinusoidally with time, at a frequency of $f$ cycles per second, between the limits $B_m$ and $-B_m$, the average rate at which heat is developed in the rod is

$$P_s = \frac{\pi^2 l r^4 (f B_m)^2}{4\rho} \qquad \text{ergs per second.}$$

(d) A solid copper rod 1 inch in diameter and 1 foot long is placed inside a long solenoid, with its center at the center of the solenoid and with its axis parallel to the axis of the solenoid. The solenoid is wound with 40 turns to the inch, and carries a current which alternates at a frequency of 60 cycles per second between a maximum value of 10 amperes in one direction and an equal value in the opposite direction. The resistivity of the rod is 1.7 microhm-centimeters. How much heat energy is developed in the rod per second? (e) How large a direct current would have to be

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established through this copper rod (parallel to its axis) to produce the same amount of heat?

*Answer.*—(d) 5.08 watts. (e) 705 amperes.

(NOTE.—In this problem no account is taken of the reaction of the magnetic field due to the current induced in the copper rod, which reaction would be appreciable.)

**119. Total Core Loss.**—Whenever the flux in iron or other magnetic substance varies with time, heat energy is developed in this core due to both magnetic hysteresis and eddy currents. The average *rate* at which heat energy is thus developed, *i.e.*, the average *power* dissipated as heat, due to both hysteresis and eddy currents is called the total "core loss." That portion of the core loss which is due to eddy currents, when the core is built up of thin sheets, is given by equation (21a). The heat *energy* per cycle due to hysteresis given by equation (16). The average *power* $P_h$ dissipated as heat due to hysteresis, namely, the *rate* at which heat energy is thus developed, is equal to the energy developed per cycle multiplied by the number of cycles per second, viz., $P_h = \eta f B_m^{1.6}$ ergs per cubic centimeter per second.

Hence the total core loss in a core having a volume of $V$ cubic centimeters, when the maximum value of the flux density has the *same value at every point* in the core, and varies sinusoidally with time, is

$$P_c = 10^{-7} V [\eta f B_m^{1.6} + \epsilon (a f B_m)^2] \qquad \text{watts} \qquad (22)$$

where

$\eta$ = the hysteresis coefficient.

$\epsilon$ = the eddy-current coefficient.

$f$ = the number of times per second that the flux passes through a complete cycle of values.

$B_m$ = the maximum value of the flux density in gausses.

The total core loss $P_c$ may be determined experimentally by any one of several different methods (see the article on *Magnetic Testing* in Pender's *Handbook for Electrical Engineers*). Since the power loss due to hysteresis varies *directly* as the frequency, and the eddy-current loss varies as the *square* of the frequency, the two losses may be separated by determining the total core loss at the two different frequencies, keeping the flux density $B_m$ constant.

It should be kept in mind in applying equation (22) that the hysteresis loss is independent of the manner in which the

flux density varies with time during each cycle of its variation, provided (a) that the corresponding hysteresis loop be a symmetrical one, (b) that the flux change sufficiently rapidly to avoid "creeping" (see Article 111), and (c) that the change in flux from its maximum value in one direction to the maximum value in the opposite direction be a *continuous* increase or a *continuous* decrease. The eddy-current loss, on the other hand, depends upon the instantaneous relation between flux and time throughout the cycle, and may be greater or less than the value given by the second term of equation (22) when this variation with time is not sinusoidal.

It should also be noted, in applying equation (22), that when the maximum flux density varies appreciably from point to point in the core, $B_m$ cannot be taken as the average value of the maximum flux density. In general, if $x$ be any quantity which has different values at different points, and $n$ be any number greater than unity, the average of $x^n$ is always *greater than* the $n$th power of the average of $x$. Consequently, when the maximum value of the flux density varies from point to point in the core, equation (22) will always give *too low* a value for both the hysteresis and the eddy-current loss. For example, were the flux density in one-half of a given core 10,000 gausses and in the other half 20,000 gausses, giving an average flux density of 15,000, the ratio of the true eddy-current loss to the eddy-current loss calculated from equation (22), using 15,000 as the value of $B_m$, is as ($\frac{1}{2} \times 1^2 + \frac{1}{2} \times 2^2$) is to ($1 \times 1.5^2$), or as 2.5 is to 2.25.

**Problem 11.**—The armature core of a certain six-pole dynamo has a volume of 2 cubic feet. The core is made up of laminations 0.0188 inch thick. The eddy-current coefficient is 0.0002 and the hysteresis coefficient 0.001. The average flux density of the magnetic field in which the core rotates is 75 kilolines per square inch. The armature rotates at a speed of 750 r.p.m.

(a) How many times does the flux linking a circuit fixed in this core pass through a complete cycle of values in 1 second? (b) Applying equation (22), and taking for $B_m$ the average flux density in gausses, what is the total power dissipated in this core due to eddy currents? (c) Due to hysteresis? (d) What is the total core loss? (e) Are these calculated losses too small or too large, and why?

*Answer.*—(a) 37.5 times per second. (b) 492 watts. (c) 678 watts. (d) 1170 watts. (e) Too small, due to nonuniformity of flux and to the assumption that the laminations are perfectly insulated from one another.

# X

## SELF AND MUTUAL INDUCTANCE

**120. General.**—Practically all apparatus and machines whose action depends upon the principle of electromagnetic induction consist essentially of two or more electric circuits, so arranged that a change in their relative position, or change in the intensity of the currents in them, induces electromotive forces in one or both of these circuits.

As shown in Article 85, the electromotive force $e$ induced in a coil by any variation in the flux which links its turns is always equal to the time rate of change of the *linkages* $\lambda$ between this flux and these turns, viz.,

$$e = \frac{d\lambda}{dt} \text{ abvolts} \tag{1}$$

where all quantities are in c.g.s. electromagnetic units. By the flux linkages $\lambda$ is meant the sum of the fluxes, say $\phi_1$, $\phi_2$, $\phi_3$, $\cdots\phi_n$ which link the separate turns of the coil, viz.,

$$\lambda = \phi_1 + \phi_2 + \phi_3 + \cdots\phi_n \tag{2}$$

In the special case when each turn is linked by the same flux, the flux linkages are

$$\lambda = N\phi \tag{3}$$

where $N$ is the number of turns in the coil.

From the fundamental relations developed in Chapter IX, it may be shown that the total flux linkages $\lambda$, of any given circuit, say No. 1, due to the current $i_1$ in this circuit and to currents $i_2$, $i_3$, etc., in any number of other circuits, is always equal to the algebraic sum of the flux linkages of this circuit due to each of these currents acting by itself, *provided* the permeability at each point of the magnetic field is independent of the value of the flux density established at this point. That is, calling $\lambda_{11}$ the flux linkages of circuit No. 1 when the current in this coil is $i_1$, and there are no currents in any other conductors in the field; $\lambda_{12}$ the flux linkages of circuit No. 1 when the current in circuit No. 2 is $i_2$, and there is no current in circuit No. 1 or in any

other conductor in the field; and so on for $\lambda_{13}$, etc., then the total flux linkages of circuit No. 1 due to all the currents in the field is

$$\lambda_1 = \lambda_{11} + \lambda_{12} + \lambda_{13} +, \text{ etc.} \tag{4}$$

When there is iron or any other ferromagnetic substance present, this relation does not hold, for the permeability at any point in such a substance depends upon the *resultant* flux density at this point. However, if the component linkages $\lambda_{11}$, $\lambda_{12}$, $\lambda_{13}$, etc., are each calculated on the assumption that the permeability at each point in the field has the value corresponding to the *resultant* flux density established at that point, then, for the particular values of the currents which produce this resultant flux density, equation (4) will give the total flux linkages of the circuit under consideration.

It may also be shown, from the relations developed in Chapter VIII, that for a given value of the permeability at each point in the field, the flux linkages of a given circuit, due to a current in this particular circuit, is directly proportional to this current. That is, the flux linkages of circuit No. 1 due to a current $i_1$ in this circuit may be written

$$\lambda_{11} = L_1 i_1 \tag{5}$$

where $L_1$ is a coefficient whose value depends upon (1) the number of complete turns made by the conductor which forms this circuit, (2) the shape, size and arrangement of these turns, and (3) the shape, size, location and permeability of every magnetic substance in the field. This coefficient $L_1$ is called the "coefficient of self-induction," or simply the "self-inductance," of the given circuit.

When the field contains no iron or other ferromagnetic substances, the self-inductance of any given electric circuit in the field is entirely independent of the value of the current in it, and also of the value of the current in any other conductor in the field.[1] On the other hand, when the magnetic field produced by the current in the given circuit lies wholly or partly in iron or

[1] This, of course, is true only for a given distribution of the current in the conductors which form the circuits. Due to the eddy currents induced in a conductor when the current in it varies rapidly with time, the self-inductance of a circuit to a rapidly varying current may be appreciably different from the self-inductance of this circuit to a current which varies slowly. This effect, however, is not as a rule appreciable in ordinary circuits for frequencies below 60 cycles per second.

other ferromagnetic substance, the self-inductance of this circuit in general depends upon the value of the current in it and also upon the value of the current in every other conductor in the field.

Similarly, for a given value of the permeability at each point in the field, the flux linkages $\lambda_{12}$ of any circuit No. 1 due to a current $i_2$ in any other circuit No. 2, is directly proportional to the current in circuit No. 2, viz.,

$$\lambda_{12} = M_{12}i_2 \tag{6}$$

where $M_{12}$ is a coefficient whose value depends upon (1) the number of complete turns made by the conductors which form each circuit, (2) the shape, size and arrangement of these turns, (3) the position of the two circuits with respect to each other, and (4) the shape, size, location and permeability of every magnetic substance in the field. This coefficient $M_{12}$ is called the "coefficient of mutual inductance," or simply the "mutual inductance" of circuit No. 2 with respect to circuit No. 1.

When the field contains no iron or other ferromagnetic substance, the mutual inductance of one circuit with respect to another is entirely independent of the values of the currents in the two circuits, and also of the value of the current in any other circuit in the field. (This is strictly true only when the eddy currents in the conductors which form the circuits are negligible; see the footnote on preceding page.) On the other hand, when the magnetic field lies wholly or partly in iron or other ferromagnetic substance, the mutual inductance of one circuit with respect to another in general depends upon the values of the currents in both circuits, and upon the value of the current in every other conductor in the field.

Brief and easily remembered definitions of these two coefficients are:

1. The self-inductance L of a circuit is the number of flux linkages of this circuit due to unit current in it.

2. The mutual inductance M of one circuit with respect to another is the number of flux linkages of the first circuit due to unit current in the second.

In applying these simple definitions to circuits wound on iron cores, it must be remembered that they apply, not to the flux linkages which would be produced by actually sending unit current through the coil in question, but to the flux linkages

which unit current would produce *if the permeability at each point remained constant at the value corresponding to the resultant flux density at this point at the particular instant of time under consideration.*

With this understanding, the total flux linkages of any given circuit, say No. 1, may always be written

$$\lambda_1 = L_1 i_1 + M_{12} i_2 + M_{13} i_3 +, \text{ etc.} \tag{7}$$

where $L_1$ is the self-inductance of this circuit, and $M_{12}, M_{13}$, etc., are the mutual inductances, with respect to this circuit, of all the other electric circuits in the field.

In this summation the positive sense of the currents $i_2$, $i_3$, etc., must be so chosen with respect to the direction of $i_1$ that, when these currents are actually in this direction, they produce through the circuit of $i_1$ a flux in the same direction as that due to $i_1$, viz., in the right-handed screw direction with respect to $i_1$. When the current in any circuit, say No. 2, is actually in the opposite direction, and $i_2$ is used as the symbol to designate this current, then the term $M_{12} i_2$ in equation (7) must be written with a negative sign in front of it (see Article 125).

From equations (1) and (7) it follows that the total electromotive force induced in any circuit, say No. 1, due either to (*a*) a variation in the value of the current in this circuit or in any other circuit, or to (*b*) a variation in the size, shape, direction or location of any one or more of a group of circuits, may always be expressed by a relation of the form

$$e_1 = \frac{d}{dt}(L_1 i_1 + M_{12} i_2 + M_{13} i_3 +, \text{ etc.}) \tag{8}$$

In the particular case when there is no iron or other ferromagnetic substance in the field, and when every circuit remains fixed in size, shape and position, the induction coefficients are all constants, and this relation becomes

$$e_1 = L_1 \frac{di_1}{dt} + M_{12} \frac{di_2}{dt} + M_{13} \frac{di_3}{dt} +, \text{ etc.} \tag{8a}$$

When the convention above stated in regard to the algebraic signs of the currents is adopted, the electromotive force $e_1$ given by either of these equations is in the *left-handed* screw direction with respect to the current $i_1$. That is, a positive value of $e_1$ in either of these equations means a *back* electromotive force with respect to the current $i_1$.

Equation (8) is a general relation of fundamental importance in the calculation of the performance of all electric devices in which a variable or alternating current exists. Several simple practical applications are given in the latter part of this chapter.

When the flux linkages $\lambda$ are expressed in practical units, viz., in volt-seconds, and the current $i$ in amperes, the corresponding unit of inductance is called the "henry." When the inductances are expressed in henries, the currents in amperes, and time in seconds, equation (8) or (8a) gives the induced electromotive force in volts. The c.g.s. electromagnetic unit of inductance, namely, flux in maxwells divided by current in abamperes, is called the "abhenry." When the inductances are expressed in abhenries, the currents in abamperes, and time in seconds, equation (8) or (8a) gives the induced electromotive force in abvolts.

The relation between the henry and the abhenry is that

$$1 \text{ henry} = 10^9 \text{ abhenries}$$

A convenient submultiple of the henry is the millihenry, which is $\dfrac{1}{1000}$ henry. Hence

$$1 \text{ millihenry} = 10^6 \text{ abhenries.}$$

**121. Self-inductance of a Coil.**—From the definitions given in the preceding article it is always possible to express the value of the self-inductance of a circuit in terms of (1) the dimensions and arrangement of the conductors which form it and of (2) the magnetic permeability of these conductors and of the surrounding medium. In the particular case of a coil of $N$ turns so arranged that each of the lines of force produced by a current in it links every turn, the value of its self-inductance may be expressed directly in terms of the number of turns in it and the reluctance $\mathcal{R}$ of the magnetic circuit of the flux which is produced when a current is established in it. Such a coil will be referred to as a coil with no magnetic leakage.

A coil with no magnetic leakage can never be absolutely realized, but is closely approximated in any concentrated winding, and also in a distributed winding, provided this winding has an iron core. Again, a long air-core solenoid may be considered as approximately a coil without leakage, provided the radial thickness of the winding is small in comparison with the internal radius of the winding.

As shown in Article 88, the total flux $\phi$ due to a current $i$ in a coil of $N$ turns, when there is no magnetic leakage, is equal to the magnetomotive force of this current, divided by the reluctance $\mathcal{R}$ of the path of the flux $\phi$. That is

$$\phi = \frac{4\pi N i}{\mathcal{R}}$$

where all quantities are in c.g.s. electromagnetic units. The number of linkages between the turns of this coil and the lines of force which represent this flux $\phi$ is then $N\phi$. Whence the flux linkages of the coil due to the current $i$ in it is

$$\lambda = \frac{4\pi N^2 i}{\mathcal{R}}$$

Hence the linkages per unit current, namely, the self-inductance of the coil, is

$$L = \frac{4\pi N^2}{\mathcal{R}} \tag{9}$$

That is, *the self-inductance of a coil with no magnetic leakage is in c.g.s. electromagnetic units, equal to $4\pi$ times the square of the number of turns in the coil, divided by the reluctance of the path of the flux which is produced by a current in it.*

It should be carefully noted that the reluctance $\mathcal{R}$ in equation (9) is always to be taken as the reluctance of the path of the flux which a given current $i$ in the coil under consideration would produce were there no other currents in its vicinity. When the magnetic field is due to currents in several different circuits, $\mathcal{R}$ is not the reluctance of the path of the total or resultant flux through the given coil, but is the reluctance of the path of the flux which would be produced by the current $i$ acting alone. When the permeability at every point in the magnetic field is constant, this reluctance $\mathcal{R}$ is a constant, and therefore the self-inductance $L$ is constant.

When the magnetic field contains iron or other ferromagnetic substance, the reluctance $\mathcal{R}$ is not constant, but depends upon the value of the current in the given coil. Moreover, when, in addition to the current in the given coil, currents also exist in other circuits in its vicinity, the reluctance $\mathcal{R}$ is to be taken as the reluctance of the path of the flux which the current in this particular coil would by itself produce in a medium whose permeability at each point is equal to that corresponding to the *resultant* flux density at this point due to *all* the currents in the field.

Hence, when there are two or more coils on a common iron core, as in a transformer, the self-inductance of each coil is a function not only of the current in that particular coil but also of the current in the other coil or coils.

To illustrate the application of equation (9), consider the simple case of a long air-core solenoid. As shown in Article 101, the total reluctance of the path of the flux produced by a current in such a winding is approximately $\dfrac{l}{S}$, where $l$ is the length of the winding and $S$ the cross-section of the space enclosed by it. Also, as shown in Article 101, except for those turns near the two ends of the winding, each turn is linked by the same number of lines of force. Therefore, from equation (9), neglecting the lateral leakage of flux near the two ends, and the flux in the space occupied by the conductors which form the turns and in
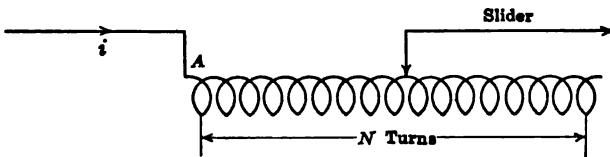


Fig. 82.

the insulation between them, the self-inductance of a long air-core solenoid is

$$L = \frac{4\pi N^2 S}{l} \qquad \text{abhenries} \qquad (10)$$

when $l$ is in centimeters and $S$ in square centimeters.

Such a solenoid is sometimes used as a variable inductance, with a slider so arranged that any desired number of turns may be inserted into the circuit, as indicated in Fig. 82. It should be noted, however, that when a solenoid is used in this manner, the inductance of that part which is in the circuit, *i.e.*, between the slider and the terminal $A$, varies *directly* as the number of active turns, and not as the square of this number of turns, except when the slider is close to the terminal $A$. This follows from the fact that as the slider is moved toward the right, both the active length $(l)$ and the number of active turns $(N)$ increase at the same rate. When the slider is close to the terminal $A$, equation (10) is of course no longer applicable, for the reluctance is then no longer $\dfrac{l}{S}$, but is practically a constant irrespective of

the length $l$.   The inductance of the active part of the coil, when $l$ is small, then varies as the square of the number of active turns. Hence, when the sliding contact is moved from $A$ to the right, bringing into the circuit more and more turns, the inductance first increases as the *square* of the distance between the slider and $A$, then less rapidly, and finally becomes directly proportional to this distance.

Exact formulas for the self-inductance of a coil can be deduced only by taking into account not only the lines of force which link each turn, but also those lines which pass through the conductor which forms the winding and those which pass between successive turns.   Such calculations are too complicated to be given here.   Exact formulas for a number of practical cases are given in the *Bulletin of the Bureau of Standards*, Vol. 8, p. 1 (1912).   A number of such formulas are also given in the article on *Inductance and Inductive Reactance* in Pender's *Handbook for Electrical Engineers.*

**Problem 1.**—A current of 5 amperes in a certain coil of one turn produces a total flux of 15,000 maxwells.   (*a*) What is the reluctance of the path of this flux?   (*b*) What is the self-inductance of this coil?   (*c*) If the coil had a concentrated winding of 10 turns instead of 1, what would be the total flux produced by a current of 5 amperes?   (*d*) What would be the reluctance of the path of this flux?   (*e*) What would be the self-inductance of this coil of 10 turns?

*Answer.*—(*a*) 0.000419 oersted.   (*b*) 0.03 millihenry.   (*c*) 150,000 maxwells.   (*d*) 0.000419 oersted.   (*e*) 3.0 millihenry.

**Problem 2.**—A solenoid 100 centimeters in length is uniformly wound with 1000 turns in one layer.   The mean diameter of the solenoid is 5 centimeters.   (*a*) What is the self-inductance of this solenoid?   (*b*) Were the solenoid cut in half, what would be the self-inductance of each half?   (*c*) If the winding of the solenoid were in two layers, each of 1000 turns and the mean diameter were kept the same (namely; 5 centimeters) what would be its self-inductance?

*Answer.*—(*a*) 2.47 millihenries.   (*b*) 1.23 millihenries.   (*c*) 9.88 millihenries.

**Problem 3.**—What is the self-inductance of the winding on the iron core described in Problem 12 of Article 90, for the particular value of the current specified?

*Answer.*—0.855 henries.

**Problem 4.**—(*a*) Referring to Fig. 59 and Problem 4 of Article 98, prove that the average value of the flux which links the current $i$ in the loop formed by the two conductors of a transmission line (two parallel wires) is

$$\phi = 4li\left[\log_e \frac{2D}{d} + \frac{\mu}{4}\right] \qquad \text{maxwells} \qquad (11)$$

where *d* is the diameter of each conductor, in centimeters, *l* the length of the line (length of each conductor) in centimeters, *D* is the distance between the centers of the two conductors in centimeters, $\mu$ is the permeability of the wire in c.g.s. electromagnetic units, and *i* is the current in each wire in abamperes. In this formula the flux produced by the current in the two ends of the loop (*i.e.*, by the current in the circuits which are connected to the two ends of the transmission line) is neglected.

(*b*) From equation (4) prove that the self-inductance of the transmission line *per centimeter length of wire* is

$$L = 2\left[ \log_\epsilon \frac{2D}{d} + \frac{\mu}{4} \right] \qquad \text{abhenries} \qquad (12)$$

(*c*) From equation (12) prove that the self-inductance *per mile of wire* in the line is

$$L = 0.741 \log_{10} \left( \frac{2D}{d} \right) + 0.0805\mu \qquad \text{millihenries} \qquad (12a)$$

where both *D* and *d* are in the same unit.

(*d*) From equation (12a) calculate the total self-inductance of a transmission line, 100 miles long, made up of two No. 0000 A. W. G. aluminum wires, when the distance between the centers of these two wires is 6 feet.

*Answer.*—(*d*) 0.386 henries.

NOTE:—Tables of self-inductance of various sizes of parallel wires, for various distances between centers are given in electrical engineering handbooks. See, for example, pp. 787 and 788 of Pender's *Handbook for Electrical Engineers*. The inductance of a stranded wire is equal, to a close approximation, to that of a solid wire which has the same cross-sectional area of metal.

## 122. Mutual Inductance of Two Coils.

—Just as the self-inductance of a coil may be expressed in terms of the number of turns in it and the reluctance of the path of the flux produced by a current in it, so may the mutual inductance of two coils be expressed in terms of the number of turns $N_1$ and $N_2$ in these two coils and a definite reluctance.

In Fig. 83 are indicated, by means of the small circles 1 and 1' and 2 and 2', two electrically independent coils. The circles represent the intersection of the windings of the two coils by a plane through the center of each. For the sake of simplicity, the figure is drawn for coils of one turn each, but in the deductions given below the two coils will be considered as having $N_1$ and $N_2$ turns respectively. In these deductions it is also assumed that the number of lines of force in the space occupied by the conductors and the insulation between turns is negligible in comparison with the number of lines of force which link all the turns of either coil.

Consider first the case when the current in coil No. 2 is $i_2$ and the current in coil No. 1 is zero. (In the particular case when there is no iron or other ferromagnetic substance in the field, and coil No. 2 has but one turn, the lines of force due to this current will have the general shape indicated in Fig. 83.) The total flux produced by the current $i_2$ will then be $\phi_2 = \dfrac{4\pi N_2 i_2}{\Re_2}$ where $\Re_2$ is the reluctance of the path of this total flux $\phi_2$. Each line of this flux will link coil No. 2, but only a portion of these
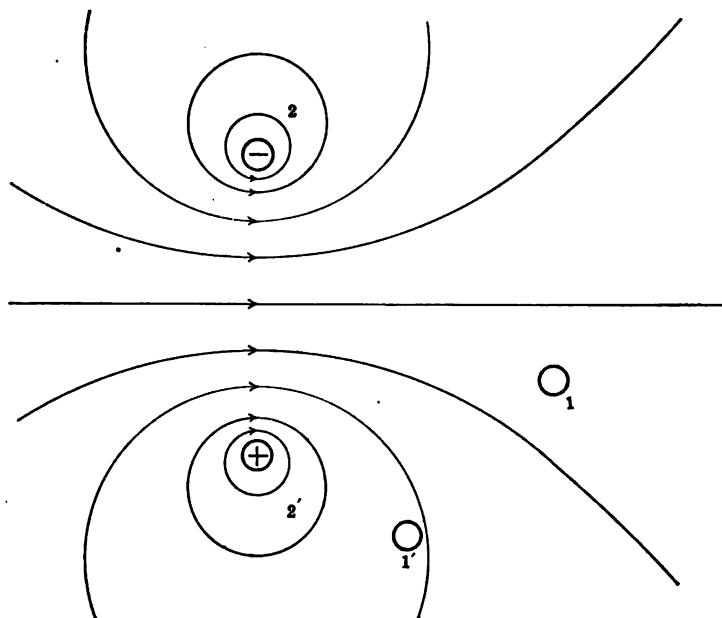


FIG. 83.

lines, say $\phi_{12}$, will link coil No. 1. (In Fig. 83 the flux $\phi_2$ is represented by 9 lines and the flux $\phi_{12}$ by 2 lines.)

Let the reluctance of the path of the flux $\phi_{12}$ which links both coils be $\Re_{12}$. Then, since by hypothesis there is no current in coil No. 1, the resultant magnetomotive force which acts on this reluctance $\Re_{12}$ is simply $4\pi N_2 i_2$. Whence

$$\phi_{12} = \frac{4\pi N_2 i_2}{\Re_{12}}$$

This is the flux which, under the conditions specified in the first paragraph of this article, links each of the $N_1$ turns of coil

No. 1. Whence the total flux linkages of coil No. 1 due to a current $i_2$ in coil No. 2 is

$$\lambda_{12} = N_1\varphi_{12} = \frac{4\pi N_1 N_2 i_2}{\mathcal{R}_{12}}$$

Consequently, the mutual inductance of coil No. 2 with respect to coil No. 1 is

$$M_{12} = \frac{4\pi N_1 N_2}{\mathcal{R}_{12}} \tag{13}$$
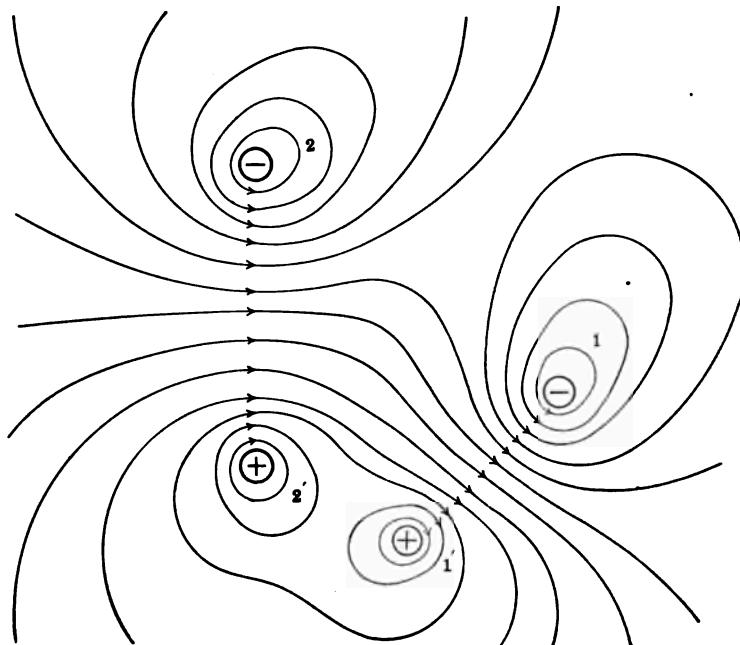


FIG. 84.

That is, *the mutual inductance of a coil 2 with respect to a coil 1 is, in c.g.s. electromagnetic units, equal to $4\pi$ times the product of the number of turns in the two coils, divided by the reluctance of that portion of the path of the flux, due to a current in No. 2 only, which links No. 1.* This formula, like equation (9), is directly applicable only when the flux in the space occupied by the winding of the coils is negligible in comparison with the total flux which links both coils.

The significance of the reluctance $\mathcal{R}_{12}$ should be carefully noted. It is the reluctance of the path of that portion of the flux which links coil No. 1 when there is no current in any other coil than

coil No. 2. When there is no iron or other ferromagnetic sub-stance in the field, this reluctance is a constant, and therefore the mutual inductance $M_{12}$ is a constant. When the coils have iron cores, the mutual inductance $M_{12}$ is not a constant, but its value depends upon that of the permeability corresponding to the resultant flux density established by the currents which may exist in the two windings.

When currents exist simultaneously in the two coils, the resultant lines of force are entirely different in shape and distri-bution from those produced by a current separately in either coil. For example, considering the same two coils as shown in Fig. 83, when a current is established in No. 1 as well as in No. 2, in the same relative direction as that in No. 2, the resultant lines of force are as indicated in Fig. 84. Were the current in either coil reversed, an entirely different distribution of lines of force would result.

The resultant lines of force which link the turns of two separate coils are usually referred to as the "mutual" flux between these coils. The reluctance of the path of this mutual flux may also be called the mutual reluctance. The value of this mutual reluctance is not a constant, but in general depends upon the relative values and directions of the currents in the two coils, even when there are no ferromagnetic substances present. A particu-lar value of this mutual reluctance is that corresponding to *no* current in coil No. 1, and it is *this particular value* of the mu-tual reluctance that must be used for the reluctance $\mathcal{R}_{12}$ in equation (13).

It is also of interest to note in this connection that in general the mutual flux between two coils bears no fixed relation to their mutual inductance. In fact, the mutual inductance may be zero, and yet when currents exist in both coils, a part of the resultant lines of force due to these two currents may link both coils. For example, two coils arranged with their planes mutually perpendicular and with the center of each coil on the axis of the other, have no mutual inductance; yet, when these two coils carry currents, some of the resultant lines of force will in general link both coils. In the particular case of a transformer, however, the mutual flux is substantially proportional to the mutual inductance (see Article 127).

Equation (13) gives the value of the mutual inductance of

coil No. 2 with respect to coil No. 1. In exactly the same manner, the mutual inductance of coil No. 1 with respect to coil No. 2 is

$$M_{21} = \frac{4\pi N_1 N_2}{\mathfrak{R}_{21}} \tag{14}$$

where $\mathfrak{R}_{21}$ is the reluctance of the path of the flux which is produced by a current in coil No. 1 when there is no current in coil No. 2 (see Fig. 85). The shape and location of the path of this
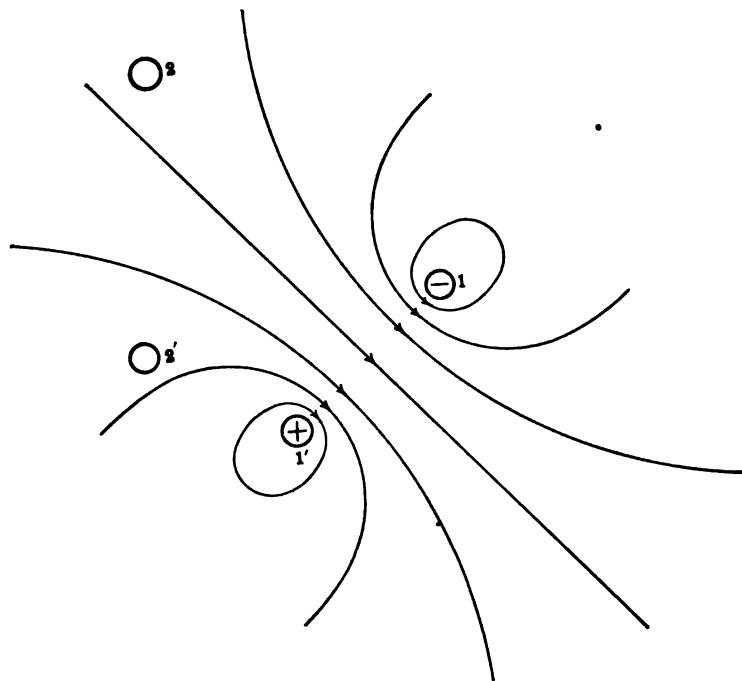


FIG. 85.

flux is in general entirely different from that of the path of the mutual flux produced by a current in coil No. 2 when there is no current in coil No. 1 (compare Figs. 83 and 85).

However, as is shown in the next article, *the mutual inductance of one of two given coils with respect to the other is always equal to the mutual inductance of the second coil with respect to the first*, viz.,

$$M_{12} = M_{21} \tag{15}$$

Comparing this relation with equations (13) and (14), it follows that, although the shape and location of the path whose reluc-

tance is $\mathfrak{R}_{12}$ may be different from the shape and location of the path whose reluctance is $\mathfrak{R}_{21}$, the numerical values of these two reluctances are always equal.

Equations (13) and (14) for the mutual inductance of two coils are based on the assumption that every line of force which links any one turn of either of the two coils links every other turn of this particular coil, *i.e.*, the flux in the space occupied by the conductors which form the turns and by the insulation between turns is neglected. It is possible, by considering each turn separately, to take this "leakage" flux into account, but the resulting formulas are too complicated to be given here. See the *Bulletin of the Bureau of Standards*, Vol. 8, p. 1 (1912) and the article on *Inductance and Inductive Reactance* in Pender's *Handbook for Electrical Engineers*.

**Problem 5.**—Two electric circuits, each of a single turn, are fixed in a definite position with respect to each other. There are no magnetic substances in their vicinity. When there is a current of 5 amperes in No. 1 and no current in No. 2, the total flux produced is 135,000 maxwells, of which 75,000 maxwells link No. 2. When there is no current in No. 1 and 10 amperes in No. 2, the total flux produced is 195,000 maxwells.

(a) What is the self-inductance of circuit No. 1? (b) Of circuit No. 2? (c) What is the mutual inductance of one circuit with respect to the other? (d) How much flux is produced through circuit No. 1 when the current in No. 2 is 5 amperes and there is no current in circuit No. 1? (e) When the current in circuit No. 1 is 8 amperes and the current in circuit No. 2 is 3 amperes, and both currents link the mutual flux in the same direction, what is the total flux through each circuit? (f) When the currents have the same values as in (e), but link the mutual flux in opposite directions, what is the total flux through each circuit?

*Answer.*—(a) 0.27 millihenry. (b) 0.195 millihenry. (c) 0.15 millihenry. (d) 75,000 maxwells. (e) 261,000 maxwells through No. 1, and 178,500 maxwells through No. 2. (f) 171,000 maxwells through No. 1, and 61,500 maxwells through No. 2.

**Problem 6.**—Referring to Problem 9 of Article 101, what is the mutual inductance between the windings of the standard solenoid there described? Calculate this inductance first from the relation that the mutual inductance of the two windings in abhenries is equal to the flux linkages of the secondary per abampere of current in the primary, and secondly from the relation that the mutual inductance is equal to $4\pi$ times the product of the number of turns in the two windings divided by the reluctance of the path of that part of the flux due to a current in the primary which links the secondary turns.

*Answer.*—0.997 millihenries.

**Problem 7.**—In Fig. 86 are shown the two wires $A$ and $B$ of a power transmission line and the two wires 1 and 2 of a telephone line which is

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parallel to, and in the same horizontal plane, as the transmission line. The distances are as indicated in figure. Consider the wires as geometrical lines.

(a) What is the mutual inductance per mile, between the transmission line and the telephone line? (b) When the current in the transmission line is 200 amperes, how much of the flux due to this current links each mile of
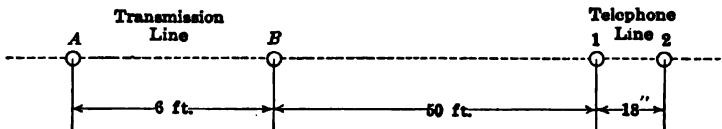


FIG. 86.

the telephone line (*i.e.*, links the loop which would be formed were the two wires of the telephone line short-circuited at points 1 mile apart)? (c) If this current has a maximum value of 200 amperes and is varying sinusoidally with time at the rate of 60 cycles per second, what is the maximum value of the electromotive force induced in the mile of telephone line.

*Answer.*—(a) 0.00100 millihenries. (b) 20,000 maxwells. (c) 0.0754 volts.

**123. Mutual Inductance and Mechanical Forces.**—Experiment shows that when a conductor which carries an electric
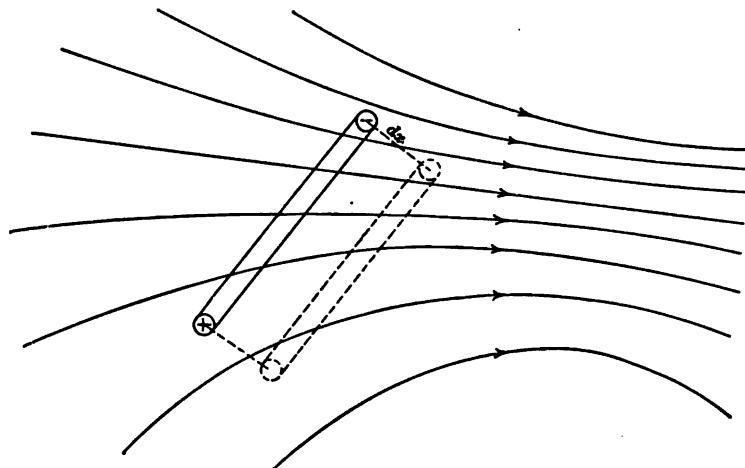


FIG. 87.

current is in a magnetic field due to any other agent (*e.g.*, another current-carrying conductor or magnet), a mechanical force is exerted on this conductor, tending to displace it with respect to this agent. A fundamental principle, from which a number of important relations may be deduced, is that the value of this force, **for a given value of the current in the given conductor, is**

always such that, when the conductor moves, **the mechanical work done by this force is equal to the amount of electric energy transferred to the conductor as a result of this motion.**

For example, consider a coil of any shape and dimensions. Let $i$ be the current in the coil, and let $\lambda$ be the number of flux linkages of the coil when it is in any given position. Let the coil be given a linear displacement $dx$ in any direction in an interval of time $dt$ (see Fig. 87). Let $d\lambda$ be the corresponding increase in the flux linkages of the coil, and let $f_x$ be the component, in the direction of $dx$, of the mechanical force which is exerted upon the coil by the agents which produce the field.

The mechanical work done by the force is then $f_x dx$. At the same time, due to the increase $d\lambda$ in the flux linkages, a back electromotive force $e = \dfrac{d\lambda}{dt}$ is induced in the coil. During the interval of time $dt$ an amount of electric energy equal to

$$eidt = \frac{d\lambda}{dt}\, idt = id\lambda$$

is therefore transferred to the coil. Equating this energy to the mechanical work $f_x dx$, it follows that the value of the force $f_x$ is

$$f_x = i\,\frac{d\lambda}{dx} \qquad \text{dynes} \qquad (16)$$

where $i$ is in abamperes and $x$ in centimeters. That is the mechanical force exerted on a coil, by the agents which produce the magnetic field in which the coil is located, has in any direction, a component equal to the product of the current in this coil by the change in the flux linkages of this coil per unit displacement of the coil in this direction.

Note that when the displacement $dx$ is such that the motion of the coil in this direction results in an *increase* in the number of lines of force which link it, the force $f_x$ has a positive value, *i.e.*, the force has a positive component in the direction of $dx$. Consequently, *the force exerted on a coil by any other coil or by magnet always tends to move the given coil into such a position that it will be linked by the maximum possible number of lines of force.*

Equation (16) is one of fundamental importance in the analysis of the mechanical forces in a magnetic field. These forces will be considered in detail in a subsequent chapter (Chapter XI). From this fundamental relation may also be deduced the important corollary that the **mutual inductance**

of one coil with respect to another is always equal to the mutual inductance of the second coil with respect to the first. To prove this, let $M_{12}$ be the mutual inductance of coil No. 2 with respect to coil No. 1, and let $i_1$ and $i_2$ be the currents in the two coils (see Fig. 88). Then the flux linkages of No. 1 due to the current $i_2$ is $\lambda_{12} = M_{12}i_2$. Imagine coil No. 1 to be displaced a distance $dx$ in any direction, the position of coil No. 2 and the current $i_2$ remaining unaltered. The change in the flux linkages of No. 1 will then be $d\lambda_{12} = i_2 dM_{12}$. Hence, from equation (16), the component, in the direction of $dx$, of the force exerted on coil No. 1 by the current $i_2$ in coil No. 2 is

$$f_x = i_1 i_2 \frac{dM_{12}}{dx} \qquad \text{dynes} \qquad (17)$$

where $i_1$ and $i_2$ are in abamperes and $M_{12}$ in abhenries, and $x$ in centimeters.
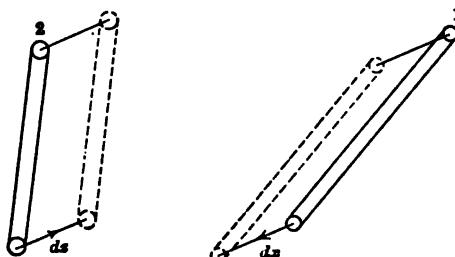
Similarly, the flux linkages of coil No. 2 due to the current $i_1$ in coil No. 1 is $\lambda_{21} = M_{21}i_1$. The change in these flux linkages when coil No. 2 is displaced a distance $dz$ in any direction, the current $i_1$ and the position of coil No. 1 remaining unaltered, is $d\lambda_{21} = i_1 dM_{21}$. Therefore the component, in the direction of $dz$, of the force exerted on coil No. 2 by the current $i_1$ in coil No. 1 is

$$f_z = i_1 i_2 \frac{dM_{21}}{dz} \qquad \text{dynes} \qquad (17a)$$

From the fundamental mechanical principle that action and reaction are equal and opposite, it follows that, when $dz$ is taken equal to $-dx$, the force $f_z$ given by equation (17a) must be equal and opposite to the force $f_x$ given by equation (17). Consequently, the change in the mutual inductance $M_{12}$ when coil No. 1 is moved a distance $dx$ must be equal to the change in $M_{21}$ when coil No. 2 is moved an equal distance $dx$ in the opposite

direction. Hence $M_{12}$ and $M_{21}$ must be equal to each other, for both of these coefficients are zero when the coils are at an infinite distance apart.

It is important, in applying equation (17), to note the positive sense of the force $f_x$. When $i_1$ and $i_2$ are both in the same direction, and $dx$ is so chosen that $dM$ is positive, *i.e.*, when the displacement $dx$ brings the coils nearer together, the force $f_x$ given by equation (17) has a positive value. This means that *two currents flowing in the same direction attract each other.* When $i_1$ and $i_2$ are in opposite directions, *i.e.*, when they link the mutual flux in opposite directions, a positive value of $dM$ gives a negative value for $f_x$. This means that *two currents flowing in opposite directions* repel each other.
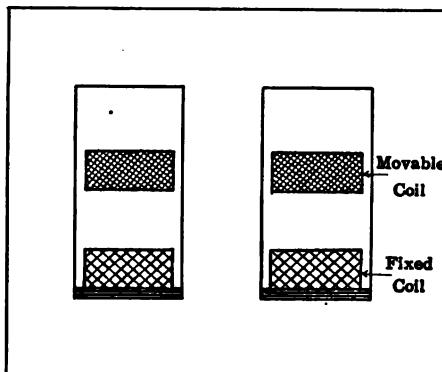
FIG. 89.

**Problem 8.**—Two coils are arranged on an iron core as indicated in Fig. 89. The lower coil is stationary and the upper one is so mounted and counterbalanced that it can move up and down. (This is the principle of the constant-current transformer.) When the upper coil is 12 inches above the lower coil, the mutual inductance of the two coils is 1 henry. When this coil is 13 inches above the lower the mutual inductance of the two coils is decreased one-half of 1 per cent.

When the currents in the two coils are respectively 30 amperes and 6 amperes and are in opposite directions, what is the average force in pounds exerted by one coil on the other during the motion of the upper coil from its first to its second position, and what is the direction of this force on the upper coil?

*Answer.*—7.96 pounds upward.

**124. Energy of the Magnetic Field Due to One or More Electric Currents in Terms of the Induction Coefficients of Their Circuits.**—In Chapter IX expressions were deduced for the energy

of a magnetic field (*a*) in terms of the magnetic flux $\phi$ and the reluctance $\Re$ of the path of this flux, and also (*b*) in terms of the flux density *B*, the magnetizing force *H*, and the volume of the space occupied by the magnetic field. The energy of the magnetic field due to one or more currents may also be expressed (*c*) in terms of the intensities of these currents and the induction coefficients of their circuits, provided there is no iron or other ferromagnetic substance in their vicinity. These expressions, which will now be deduced, are also approximately correct even when the coils do have iron cores.

Consider first a single coil by itself. Let *L* be the self-inductance of this coil, and let *i* be the current in it at any instant, and let $\dfrac{di}{dt}$ be the rate of increase of this current at this instant. Then the flux linkages of the circuit of this current at this instant is $\lambda = Li$. When *L* is constant, *i.e.*, when there are no ferromagnetic substances present, the rate of increase of these flux linkages is $\dfrac{d\lambda}{dt} = L\dfrac{di}{dt}$, which in turn is equal to the *back* electromotive force *e* induced in the circuit due to the increase of the current in it. To produce the increase *di* in the current therefore requires a transfer of

$$dW = eidt = Lidi$$

units of energy from the electric circuit to the magnetic circuit of the flux produced by this current.

The total energy stored in the magnetic field during the increase of the current from zero to any value *i* is then the integral of *Lidi* between the limits 0 and *i*, viz.,

$$W = \frac{1}{2}Li^2 \tag{18}$$

In this expression, when *L* is in henries and *i* in amperes, the energy *W* is in joules, or watt-seconds. When *L* is in abhenries and *i* in abamperes, the energy *W* is in ergs.

It is of interest to compare this expression with that for the kinetic energy of a mass *M* moving with a velocity *V*, viz.,

$$\text{Kinetic energy} = \frac{1}{2}MV^2$$

Comparing this expression with equation (18), and noting that the intensity *i* of an electric current is equal to the rate or *velocity of flow* of electricity, it is evident that the self-inductance of the

circuit of an electric current may be looked upon as a measure of the "effective mass," or *inertia*, of the electricity in this circuit. From the same point of view, the magnetic energy of an electric current may be considered as the "electrokinetic" energy of the moving electricity which constitutes the current. There is a fundamental difference, however, between the inertia of a mass $M$ and the "inertia" of the electricity in a circuit, in that the inertia of a mass $M$ is property of this particular mass, whereas the inertia of a given quantity of electricity in motion depends upon the shape and size of the circuit in which this flow takes place, and upon the nature of the surrounding medium.

Consider next two coils whose self-inductances are $L_1$ and $L_2$ and whose mutual inductance is $M$. Let $i_1$ and $i_2$ be the currents in these coils at any instant, in the *same* direction relative to the mutual flux which links them both. The flux linkages of the two coils at this instant are then respectively

$$\lambda_1 = L_1 i_1 + M i_2$$
$$\lambda_2 = L_2 i_2 + M i_1$$

When $i_1$ and $i_2$ vary with time, and $L_1$, $L_2$ and $M$ are *constants* (*i.e.*, no ferromagnetic substances present), the back electromotive forces due to the variations in these linkages are

$$e_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$
$$e_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

Hence, in time $dt$ the total energy transferred to the magnetic field of the two currents is

$$\begin{aligned} dW &= e_1 i_1 dt + e_2 i_2 dt \\ &= L_1 i_1 di_1 + L_2 i_2 di_2 + M(i_1 di_2 + i_2 di_1) \\ &= L_1 i_1 di_1 + L_2 i_2 di_2 + M d(i_1 i_2) \end{aligned}$$

The total energy $W$ stored in the magnetic field of the two currents, when they increase from zero to the values $i_1$ and $i_2$ respectively, is the integral of this expression between the limits $i_1 = 0$, $i_2 = 0$ and $i_1 = i_1$, $i_2 = i_2$. This integral is

$$W = \frac{1}{2} L_1 i_1^2 + M i_1 i_2 + \frac{1}{2} L_2 i_2^2 \tag{18a}$$

In an exactly similar manner it may be shown that when the coils are so placed that the currents $i_1$ and $i_2$ in them link the

mutual flux in *opposite directions,* the total energy of their magnetic field is

$$W = \frac{1}{2} L_1 i_1^2 - M i_1 i_2 + \frac{1}{2} L_2 i_2^2 \qquad (18b)$$

In general, the total energy of the magnetic field due to currents in any number of coils, when there are no ferromagnetic substances present, may be written

$$W = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 + \frac{1}{2} L_3 i_3^2 +, \text{ etc.}$$
$$\pm M_{12} i_1 i_2 \pm M_{13} i_1 i_3 \pm M_{23} i_2 i_3, \pm \text{ etc.} \qquad (18c)$$

where the $L$'s are their self-inductances and the $M$'s the mutual inductances of the various pairs of coils. The plus sign is to be used before an $M$-term when the two currents link their mutual flux in the same direction, and a minus sign when the two currents link the mutual flux in opposite directions.

**Problem 9.**—The currents in two air-core coils, whose total self-inductances are 2 millihenries and 5 millihenries, are maintained constant at 20 and 30 amperes respectively. The mutual inductance of these two coils is 3 millihenries.

(a) What is the energy of the magnetic field of these two currents when their directions are such that they tend to produce a magnetic flux in the same direction? (b) In opposite directions? (c) What would be the energy of their magnetic field were the center of one coil on the axis of the other and were their axes perpendicular?

*Answer.*—(a) 4.45 joules. (b) 0.85 joules. (c) 2.65 joules.

**125. Leakage Inductance and Exciting Current.**—As noted in Article 122, the path of the mutual flux produced by currents in *both* of two coils is in general different from the path of the mutual flux when there is a current in only *one* of the coils, and the reluctances of these two paths are in general different. However, when the two coils have a common iron core, as in a transformer (see Fig. 90), the path of the mutual flux is confined practically to the iron core, irrespective of the relative values of the currents in the two coils. The reluctance of the path of this flux is then practically the reluctance of the iron core.

In Fig. 90, the lines of force representing the mutual flux are shown as full lines marked $\phi_{12}$, and the lines of force which link one coil but not the other are shown as dotted lines, marked $\phi_{11}$ and $\phi_{22}$. Another arrangement of two coils on a common iron core is shown in Fig. 53, in which the mutual flux is marked $\phi_m$.

When the mutual flux is confined to a definite limited space, as in the core of a transformer, the mutual inductance of the two coils, for any given values $i_1$ and $i_2$ of the currents in them, is

$$M = \frac{4\pi N_1 N_2}{\mathfrak{R}_{12}} \tag{19}$$

where $\mathfrak{R}_{12}$ is the reluctance of this path (*e.g.*, the core) to the *resultant* flux established in it by these two currents (see Article 122).

In a transformer the currents in the two windings are in such relative directions that, for the greater part of each cycle of variation in their values, they link the mutual flux in *opposite*
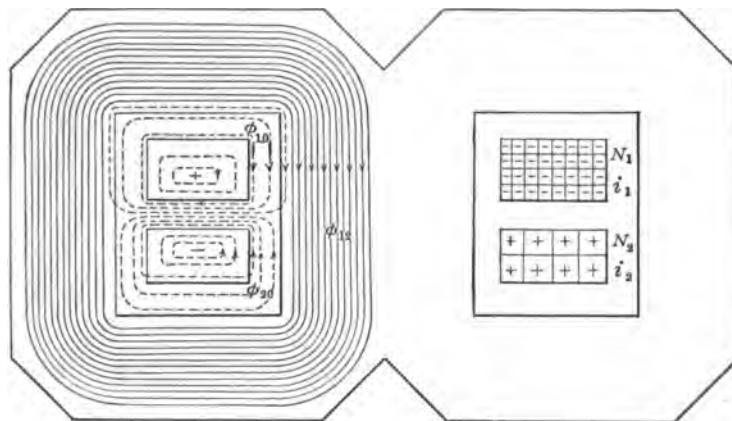


This half of section shows the lines of force only.

This half of section shows the windings only.

Fig. 90.—Cross-section of a "Shell-type" transformer perpendicular to the conductors which form the two windings.

directions. Moreover, the mutual flux is, during the greater part of each cycle of its variation, in the right-handed screw direction with respect to the primary current $i_1$. It is therefore convenient to consider the secondary current $i_2$ as having the same algebraic sign as the primary current $i_1$ when this secondary current links the mutual flux in the direction *opposite* to that of the primary current.

When this convention is adopted, the resultant current-turns which link the mutual flux is $(N_1 i_1 - N_2 i_2)$. These current-turns are equivalent to the current-turns which would be produced by a current

$$i_e = \frac{N_1 i_1 - N_2 i_2}{N_1} = i_1 - \left(\frac{N_2}{N_1}\right) i_2 \tag{19a}$$

in the primary coil alone, with no current in the secondary coil, provided this current $i_e$ produces in the core a flux which has the same *distribution* as that due to the two currents $i_1$ and $i_2$ acting jointly. In a practical transformer this provision is substantially fulfilled.

The current $i_e$ defined by equation (19), namely, *the current which is equal to the actual primary current, less the actual secondary current multiplied by the ratio of the secondary to the primary turns, is called the "exciting current" of the transformer referred to the primary.* As a first approximation, usually sufficiently close for all practical purposes in a properly designed transformer, the exciting current is equal to the current which would be taken by the primary winding were the secondary circuit open ($i_2 = 0$), viz., to the "no-load" primary current.

The mutual flux $\phi_{12}$, namely, the lines of force which link *both* windings of a transformer, for any values $i_1$ and $i_2$ of the primary and secondary currents, may then be written

$$\phi_{12} = \frac{4\pi (N_1 i_1 - N_2 i_2)}{\mathcal{R}_{12}} = \frac{4\pi N_1 i_e}{\mathcal{R}_{12}}$$

The flux linkages of the primary winding due to this mutual flux are then

$$\lambda_{12} = N_1 \phi_{12} = \frac{4\pi N_1^2}{\mathcal{R}_{12}} i_e = \frac{N_1}{N_2} \left( \frac{4\pi N_1 N_2}{\mathcal{R}_{12}} \right) i_e$$

and the flux linkages of the secondary winding due to this mutual flux are

$$\lambda_{21} = N_2 \phi_{12} = \left( \frac{4\pi N_1 N_2}{\mathcal{R}_{12}} \right) i_e$$

Note that $\left( \dfrac{4\pi N_1 N_2}{\mathcal{R}_{12}} \right)$ is equal to the mutual inductance between the two windings, as given by equation (19). Hence the flux linkages of the two windings due to the mutual flux $\phi_{12}$ may be written respectively

$$\lambda_{12} = \left( \frac{N_1}{N_2} \right) M i_e \tag{20}$$

$$\lambda_{21} = M i_e \tag{20a}$$

The flux linkages $\lambda_{12}$ of the primary are in the right-handed screw direction with respect to the primary current $i_1$. The flux linkages $\lambda_{21}$ of the secondary, however, are in the *left-handed* screw direction with respect to the secondary current $i_2$, since the positive sense of $i_2$ is taken opposite to that of $i_1$ and $i_e$.

Referring to Fig. 90, it is evident that in addition to the mutual flux $\phi_{12}$, the primary winding is linked by a certain number of lines of force which do not link the secondary winding. *The flux which links the primary winding only is called the "primary leakage flux,"* and may be designated $\phi_{10}$. Similarly, *the flux which links the secondary winding only is called the secondary leakage flux,* and may be designated $\phi_{20}$. Let $\mathcal{R}_{10}$ and $\mathcal{R}_{20}$ be the reluctances of the paths of these leakage fluxes.

As indicated in Fig. 90, some of the lines of force which represent the leakage fluxes (shown as dotted lines in the figure) pass through the space occupied by the conductors which form the windings and by the insulation between the successive turns and layers of the windings. Were all the lines which represent the primary leakage flux linked by all the primary turns, the value of this flux would be simply $\dfrac{4\pi N_1 i_1}{\mathcal{R}_{10}}$, for this flux would be due to a magnetomotive force $4\pi N_1 i_1$ acting on a reluctance $\mathcal{R}_{10}$. However, due to the fact that some of these lines link less than all the turns of this winding, the actual value of the primary leakage flux is

$$\phi_{10} = \frac{4\pi(k_1 N_1)i_1}{\mathcal{R}_{10}}$$

where $k_1$ is a factor less than unity. Or, looked at from another point of view, with respect to the primary leakage flux, the primary winding is equivalent to a coil of $(k_1 N_1)$ turns.

Similarly, with respect to the secondary leakage flux, the secondary winding is equivalent to a coil of $(k_2 N_2)$ turns, where $k_2$ is a factor less than unity. The secondary leakage flux is then

$$\phi_{20} = \frac{4\pi(k_2 N_2)i_2}{\mathcal{R}_{20}}$$

The flux linkages of the primary due to the primary leakage flux is then

$$\lambda_{10} = (k_1 N_1)\phi_{10} = \frac{4\pi(k_1 N_1)^2 i_1}{\mathcal{R}_{10}}$$

and the flux linkages of the secondary due to the secondary leakage flux is

$$\lambda_{20} = (k_2 N_2)\phi_{20} = \frac{4\pi(k_2 N_2)^2 i_2}{\mathcal{R}_{20}}$$

The quantity

$$L_{10} = \frac{4\pi(k_1 N_1)^2}{\mathcal{R}_{10}} \tag{21}$$

which represents the flux linkages of the primary, per unit of primary current, *due solely to the flux which links the primary only*, is called the "primary leakage inductance." Similarly, the quantity,

$$L_{20} = \frac{4\pi (k_2 N_2)^2}{\mathfrak{R}_{20}} \tag{21a}$$

which represents the flux linkages of the secondary, per unit of secondary current, *due solely to the flux which links the secondary ony*, is called the "secondary leakage inductance."

The leakage inductance of a winding must not be confused with the *total* self-inductance of this winding. The total self-inductance of a winding is that quantity by which the current in this winding must be multiplied in order to give the *total* flux linkages of this winding when there are no other currents present (see Article 121). The *leakage* inductance of a winding, on the other hand, is that quantity by which the current in this winding must be multiplied in order to give *that portion* of the total flux linkages of this winding due to those lines of force which link this particular winding *and no other*. The leakage inductance of a winding when there is any other winding in its vicinity is *always less than its total self-inductance.*

In terms of the leakage inductances $L_{11}$ and $L_{22}$, the flux linkages of the primary and secondary windings of a transformer, due respectively to the primary and secondary leakage fluxes, may then be written

$$\lambda_{10} = L_{10}i_1 \tag{22}$$

$$\lambda_{20} = L_{20}i_2 \tag{22a}$$

These linkages are each in the right-handed screw direction with respect to the corresponding currents.

From equations (20) and (20a) and (22) and (22a), the *total* flux linkages of the primary and secondary windings, in the right-handed screw direction respectively with respect to the currents $i_1$ and $i_2$ in these windings, may then be written

$$\lambda_1 = L_{10}i_1 + \left(\frac{N_1}{N_2}\right) M i_e \tag{23}$$

$$\lambda_2 = L_{20}i_2 - M i_e \tag{23a}$$

In these expressions the current $i_e$, namely, the exciting current, has the value (see equation 19a),

$$i_e = i_1 - \left(\frac{N_2}{N_1}\right) i_2 \tag{23b}$$

In terms of the total self-inductances $L_1$ and $L_2$ of the two windings and the mutual inductance $M$, the total flux linkages of the two windings may be written

$$\lambda_1 = L_1 i_1 - M i_2 \qquad \qquad (24)$$

$$\lambda_2 = L_2 i_2 - M i_1 \qquad \qquad (24a)$$

In these equations $L_1$ is the number of flux linkages of the primary which would be produced by unit current in this winding and no current in the secondary winding, were the permeability of the core constant at the value corresponding to the *resultant* flux density which the two currents jointly establish in the core (see Articles 121). Similarly, $L_2$ is the number of flux linkages of the secondary which would be produced by unit current in the secondary winding and no current in the primary winding, were the permeability of the core constant at this same value. The coefficient $M$ is the number of flux linkages of the primary which would be produced by unit current in the secondary were there no current in the primary (or conversely), were the permeability of the core is constant at this same value. The negative sign before $M i_2$ and $M i_1$ is used since the two currents are considered as having the same algebraic sign when they link the mutual flux in opposite directions.

On the assumption that the mutual flux is confined to the iron core, and that its distribution is independent of the relative values of the currents in the two windings (which is substantially in accord with the facts in the case of a transformer with a closed iron core), the relation between the total self-inductances $L_1$ and $L_2$ and the leakage inductances $L_{10}$ and $L_{20}$ may be readily determined as follows. Substitute in (23) and (23a) the value of $i_s$ given by (23b). There results

$$\lambda_1 = \left[ L_{10} + \left( \frac{N_1}{N_2} \right) M \right] i_1 - M i_2$$

$$\lambda_2 = \left[ L_{20} + \left( \frac{N_2}{N_1} \right) M \right] i_2 - M i_1$$

Comparing these expressions with (24) and (24a), it is evident that

$$L_1 = L_{10} + \left( \frac{N_1}{N_2} \right) M \qquad \qquad (25)$$

$$L_2 = L_{20} + \left( \frac{N_2}{N_1} \right) M \qquad \qquad (25a)$$

That is, the leakage inductance $L_{10}$ of the primary is equal to its total inductance less the product of the mutual inductance of the two windings by the ratio of the primary to the secondary turns. Similarly, the leakage inductance $L_{20}$ of the secondary is equal to its total inductance less the product of the mutual inductance by the ratio of the secondary to the primary turns.

In a transformer with a closed iron core the reluctance of the path of the mutual flux, which is entirely in iron, is only a very small fraction of the reluctance of the path of the leakage flux, which is largely in air (see Fig. 90). Hence the leakage flux is but a small fraction of the total flux, and therefore the leakage inductance of each winding is only a very small fraction of the total inductance of this winding. In other words, $L_1$ is very nearly equal to $\left(\dfrac{N_1}{N_2}\right)M$, and $L_2$ is very nearly equal to $\left(\dfrac{N_2}{N_1}\right)M$.

In the limiting case when there is no leakage flux, *i.e.*, when all the lines of force due to the currents in both windings link every turn of each coil, $L_1 = \left(\dfrac{N_1}{N_2}\right)M$ and $L_2 = \left(\dfrac{N_2}{N_1}\right)M$. Under these conditions $L_1L_2 = M^2$, or $M = \sqrt{L_1L_2}$. When there is leakage, the mutual inductance $M$ must always be less than $\sqrt{L_1L_2}$. The ratio of the mutual inductance $M$ to the square root of the product of the two total inductances $L_1$ and $L_2$, viz., the ratio

$$K = \frac{M}{\sqrt{L_1L_2}} \tag{26}$$

is called the "coefficient of coupling" of the two windings. The maximum degree of coupling, corresponding to no leakage, is unity.

The two methods of expressing the total flux linkages of each of two coils, as given respectively by equations (23) to (23b) and (24) to (24a), are equally valid representations of the facts, provided the exact significance of the coefficients is kept clearly in mind. Which set of formulas is to be used in any particular case is solely a matter of convenience.

When there is no iron or other ferromagnetic substance present, the total self-inductances $L_1$ and $L_2$ and the mutual inductance $M$ are all three independent of the values of the currents. On the other hand, except, when the two coils are closely coupled, so that the leakage fluxes are relatively small, the leakage inductances $L_{10}$ and $L_{20}$ are not constants, but depend upon the values

of these currents. Hence when there are no ferromagnetic substances present, equations (24) and (24a) are the most convenient to use.

When the coils have iron cores, or when there is any ferromagnetic substance in their vicinity, none of the induction coefficients are strictly independent of the values of the currents in the two coils. This is due to the fact that the values of these currents determine the resultant flux density in the cores, and this in turn determines the permeability of the cores. However, when the coils are on a common closed iron core, as in a transformer, the two leakage inductances are substantially independent of the values of the currents. This arises from the fact that these leakage inductances correspond to fluxes whose paths are largely in a non-magnetic region of definite dimensions (*e.g.*, the winding space occupied by the coil and the air space immediately in the vicinity of the conductors). Under these conditions equations (23) and (23b) are the more convenient expressions for the flux linkages of the two coils. In fact, this method of expressing the flux linkages is almost invariably adopted when dealing with an iron-core transformer.

Although the leakage inductances of the windings of an iron core are substantially constant, the mutual inductance of these two windings is not constant, but depends at each instant upon the particular value of the flux density and magnetizing force established in the core at this instant. This variation has an appreciable influence upon the practical performance of the apparatus. For a further discussion of the transformer, see Article 127.

**Problem 10.**—In a certain transformer the reluctances of the paths of the leakage fluxes which link the primary and secondary windings are equal to each other and each is equal to 0.15 oersted. The primary winding has 180 turns and the secondary 9 turns. When the current in the primary winding is 45 amperes, the current in the secondary winding is 860 amperes, and links the core in the direction opposite to that in which the core is linked by the primary current. The reluctance of the core of the transformer to the mutual flux is 0.00012 oersted. (These data are taken from tests on a 100-kilowatt alternating-current transformer.)

Calculate: (a) the exciting current referred to the primary, (b) the mutual inductance of the two windings, (c) the leakage inductance of the primary winding, (d) the leakage inductance of the secondary winding, (e) the total self-inductance of the primary winding, (f) the total self-inductance of the secondary winding, (g) the mutual flux, (h) the primary leakage flux,

(*i*) the secondary leakage flux, (*j*) the total flux which links the primary, (*k*) the resultant flux which links the secondary, (*l*) the total flux linkages of the primary, (*m*) the total flux linkages of the secondary. (*n*) What percentage of the total primary self-inductance is the primary leakage inductance? (*o*) What percentage of the total flux which links the primary is the primary leakage flux? (*p*) What percentage of the total secondary self-inductance is the secondary leakage inductance? (*q*) What percentage of the total flux which links the secondary is the secondary leakage flux?

*Answer.*—(*a*) 2.0 amperes. (*b*) 0.1697 henries. (*c*) 0.0271 henries. (*d*) 0.00000678 henries. (*e*) 3.42 henries. (*f*) 0.00849 henries. (*g*) 3,770,000 maxwells. (*h*) 67,800 maxwells. (*i*) 64,800 maxwells. (*j*) 3,383,000 maxwells. (*k*) 3,705,000 maxwells. (*l*) 690,800,000 maxwells. (*p*) 0.08 per cent. (*q*) 1.75 per cent.

**126. Establishment and Decay of a Current in a Circuit Containing Resistance and Inductance Only.**—Fig. 91 represents diagrammatically a circuit of total resistance *r* and total self-induc-
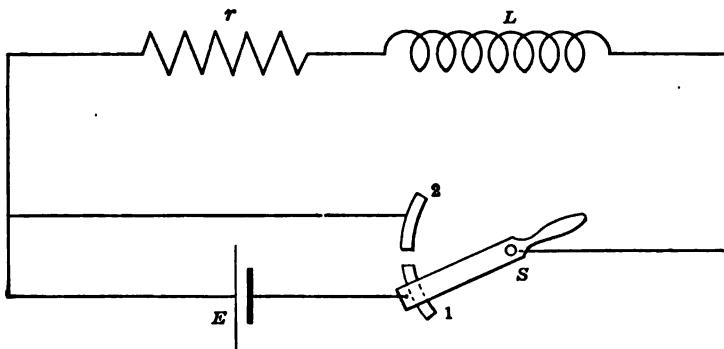


FIG. 91.

tance *L*, containing a battery whose electromotive force is *E* and a switch *S*. In such diagrams, for the sake of simplicity, the resistance and inductance are usually shown separately and "lumped." It should be clearly kept in mind, however, that each part of the circuit has a definite resistance, and that each part is linked by a definite number of lines of force, and therefore contributes to the total self-inductance. The major portion of the self-inductance, however, is due to those parts of the circuit which consist of coils of wire.

Let the circuit be initially open, *i.e.*, let the blade of the switch *S* make contact with neither of the two segments 1 and 2. Now let the blade be moved into the position indicated in the figure. A flow of electricity will immediately begin, and its rate of flow, *i.e.*, the intensity of the current, will increase until the total

resistance drop $ri$ is equal to the electromotive force $E$ of the battery.

As the current increases, however, the flux produced by it also increases, and this increase of flux induces in this circuit an electromotive force equal to the rate of change of the total flux linkages of the circuit, viz., equal to

$$e = \frac{d}{dt}(Li) \qquad (27)$$

This electromotive force is in the *opposite* direction to that of the current (see Article 82), and therefore opposite to the electromotive force of the battery. Hence at any instant $t$ seconds after closing the switch, the resultant electromotive force in the circuit is $E - e$, and this resultant electromotive force must be equal to the resistance drop $ri$ at this instant. Whence, at any instant,

$$E - \frac{d}{dt}(Li) = ri \qquad (28)$$

This relation is a perfectly general one, provided $E$ is taken to represent the resultant of all the electromotive forces in the given circuit, other than the electromotive force induced by the flux which is produced by the current in this particular circuit. When the self-inductance $L$ is constant (*i.e.*, no ferromagnetic substances present), equation (28) may be written

$$E = ri + L\frac{di}{dt} \qquad (28a)$$

In order to find the value of the current $t$ seconds after closing the switch, rearrange the terms of this equation thus:

$$\frac{d(E - ri)}{E - ri} = -\frac{r}{L}dt \qquad (28b)$$

and integrate both sides between the limits $t = 0$ and $t = t$. Note that in the special case under consideration $i = 0$, when $t = 0$, and $i = i$ when $t = t$. Integration gives, when $E$ is constant,

$$\log_\epsilon \left(\frac{E - ri}{E}\right) = -\frac{rt}{L}$$

or

$$\frac{E - ri}{E} = \epsilon^{-\frac{rt}{L}}$$

where $\epsilon$ is the base of the natural system of logarithms, viz., $\epsilon = 2.718$. Whence

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$$i = \frac{E}{r}(1 - \epsilon^{-\frac{rt}{L}}) \qquad (29)$$

(For tables of values of $\epsilon^{-x}$, where $x$ is any number, see article on *Exponential Functions* in Pender's *Handbook for Electrical Engineers*).

The physical interpretation of this equation is that the current reaches its steady value $I = \frac{E}{r}$ only after the time $t$ measured from the instant of closing the circuit has become sufficiently great to make the term $\epsilon^{-\frac{rt}{L}}$ sensibly equal to zero. However, when the ratio $\frac{L}{r}$ is small, as is usually the case, this term $\epsilon^{-\frac{rt}{L}}$ becomes practically zero for $t$ equal to a small fraction of a second, and the current therefore reaches its steady value $I = \frac{E}{r}$ almost immediately after the circuit is closed. When the self-inductance is large, as in the case of a coil of a great number of turns wound on a core of high permeability, the ratio $\frac{L}{r}$ may be relatively large, in which case several seconds may elapse before the current reaches its steady value.

In any case, after an interval of time $T = \frac{L}{r}$, the current is

$$i_r = \frac{E}{r}(1 - \epsilon^{-1}) = \frac{E}{r}\left(1 - \frac{1}{2.718}\right) = 0.632\frac{E}{r}$$

That is, after an interval of time $T = \frac{L}{r}$, the current reaches 63.2 per cent. of its final value. This interval of time $T = \frac{L}{r}$ is called the "time constant" of the circuit

The relation between current and time given by equation (29) may be represented graphically by plotting the current $i$ as the ordinates against the time $t$ as abscissas. The curve has the general shape shown by curve $A$ in Fig. 92, that is, the current rises rapidly at first and then more slowly, becoming asymptotic to the line corresponding to $I = \frac{E}{r}$, which represents the final steady value of the current. The abscissa $T$ of the point whose ordinate is $i = 0.632I$ is equal to the time constant of the circuit.

Equation (29) holds only when the inductance of the circuit is *constant*, and is therefore not strictly applicable to a circuit which

contains a coil with an iron core. However, when for $L$ is taken the average self-inductance of such a circuit, equation (29) will give the relation between current and time to a rough degree of approximation.

It must always be kept in mind, in applying equation (29), that the $L$ in this equation is the *total* self-inductance of the circuit, and the $r$ is the *total* resistance of the circuit. For
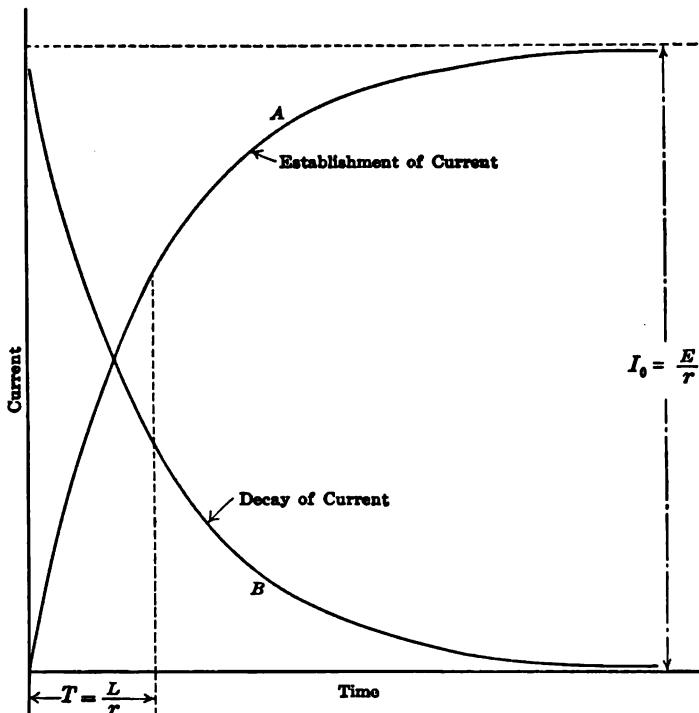


FIG. 92.—Rise and fall of current in an inductive circuit.

example, when the source of the electromotive force $E$ is a direct-current generator, the self-inductance of the generator must be included as part of the total self-inductance $L$.

When the current in the circuit at time $t = 0$ has a value $I_0$, say, the left-hand member of equation (28b) must be integrated between the limits $i = I_0$ and $i = i$. The value of the current at time $t$ is then

$$i = \frac{E}{r}\left(1 - \epsilon^{-\frac{rt}{L}}\right) + I_0\epsilon^{-\frac{rt}{L}} \qquad (29a)$$

In particular, when the source of the electromotive force $E$ is short-circuited at a given instant, and then removed from the circuit, by shifting the blade of the switch $S$ in Fig. 91 from the contact 1 to the contact 2, the current in the circuit $t$ seconds after this change is made will be

$$i = I_0\, \epsilon^{-\frac{rt}{L}} \qquad\qquad (29b)$$

The current will therefore not fall to zero immediately. Instead, the flow of electricity in the circuit will continue for an appreciable time, but the rate of flow (*i.e.*, the intensity of the current) will rapidly decrease, falling to 36.8 per cent. of its initial value in time $T = \dfrac{L}{r}$. The change of the current with time under these conditions is shown by curve $B$ in Fig. 92.

The effect just described is due to the fact that when the current decreases, the flux linking it in the right-handed screw direction *decreases*, and therefore an electromotive force (numerically equal to $L\dfrac{di}{dt}$) is induced in the circuit in the *same* direction as the current, thereby tending to maintain the current. Or, looked at from another point of view, the energy which is stored in the magnetic field of the current when the current is established, is returned to the circuit when the current dies out, and is sufficient to supply, for an appreciable interval of time, the energy which is dissipated as heat in the resistance of the circuit.

Problem 11.—The primary winding of the transformer described in Problem 10 of Article 125 is connected in series with a 110-volt direct-current generator and a switch $S$. The resistance of this winding is 1.5 ohms. The internal resistance and inductance of the generator and the resistance and inductance of the leads are negligible in comparison with the resistance and inductance of the primary winding of the transformer. The secondary winding of the transformer is left open-circuited. Assume the induction coefficients of the transformer to be constant, at the values found in Problem 9.

(a) What is the time constant of the primary winding of the transformer? (b) What will be the final steady value of the current established in this winding when the switch is closed? (c) How long will it take the current to reach 63.2 per cent. of this final value? (d) What is the value of the voltage induced in this winding at the instant the switch is closed? (e) What is the value of this induced electromotive force when the current is equal to 63.2 per cent. of its final value? (f) Draw, to scale, curves showing the value of the current and the electromotive force induced in this winding for the first 10 seconds after closing the switch. (g) What is the value of the electromotive force induced in the secondary winding when the current

in the primary winding is 63.2 per cent. of its final value? (*h*) If the second-ary winding had 100 times as many turns as the primary winding, what would be the value of the secondary induced electromotive force at this instant? (*i*) If 3 seconds after closing the switch, the generator were short-circuited, how long would the current in the primary coil take to fall to 5 amperes?

*Answer.*—(*a*) 2.28 seconds. (*b*) 73.4 amperes. (*c*) 2.28 seconds. (*d*) 110 volts. (*e*) 40.5 volts. (*g*) 2.02 volts. (*h*) 4050 volts. (*i*) 5.39 seconds.

**127. Alternating-current Transformer.**—As has already been noted, two coils wound on a common iron core, as shown digram-matically in Fig. 90, constitute what is known as an alternating-current transformer. Such a device is also called a "static" transformer, since there are no moving parts in it. In general, any two stationary electric circuits, so arranged that a current in one produces a magnetic flux through the other, may be called a transformer. When there is no magnetic core the combination of the two coils is usually called an "air-core" transformer.

The principal use of a static transformer is to convert electric power at one voltage into electric power at another voltage. Since there are no moving parts to the apparatus, such a conver-sion of power can be produced by it only when the currents in the two windings *vary with time.* It is the ease with which the power of an alternating current can be "stepped up" or "stepped down" to any desired voltage, by means of such a simple device, that has led to the common use of alternating currents for lighting and industrial purposes.

Consider first the ideal case of two coils of negligible resistance, and also of negligible *leakage* inductance, *i.e.*, so arranged that *all* the flux which links one coil links the other. Let $\phi$ be this flux, and let $N_1$ be the number of turns in one coil and $N_2$ the number of turns in the other. The flux linkages of the two coils are then respectively $N_1\phi$ and $N_2\phi$, and the electromotive forces induced in them when this flux varies by an amount $d\phi$ in $dt$ sec-onds are

$$e_1 = N_1\frac{d\phi}{dt} \qquad \text{and} \qquad e_2 = N_2\frac{d\phi}{dt}$$

Let the first coil to be connected to a source of electromotive force (*e.g.*, an alternating-current generator) which establishes between its terminals a *varying* potential difference $v_1$, and let the second coil be left open, *i.e.*, no external circuit connected to its terminals. The potential difference $v_1$ impressed across the termi-

nals of the first coil will establish a varying current $i_{10}$ in this coil, and the flux $\phi$ produced by this current will link both coils, inducing in them electromotive forces $e_1 = N_1 \dfrac{d\phi}{dt}$ and $e_2 = N_2 \dfrac{d\phi}{dt}$ respectively.

The direction of each of these electromotive forces will bear a *left-handed* screw relation to the direction of the flux $\phi$, and since the flux $\phi$ bears a *right-handed* screw relation to the current $i_{10}$, the electromotive force $e_1 = N_1 \dfrac{d\phi}{dt}$ will be a *back* electromotive force with respect to this current. Moreover, since by hypothesis the resistances and leakage inductances of the two coils are negligible, this back electromotive force must be equal to the voltage $v_1$ impressed on the terminals of the primary, viz.,

$$e_1 = N_1 \frac{d\phi}{dt} = v_1$$

Similarly, the electromotive force $e_2$ induced in the secondary winding will be

$$e_2 = N_2 \frac{d\phi}{dt} = \frac{N_2}{N_1} v_1$$

and this in turn will be equal to the potential difference $v_2$ between the terminals of the secondary winding. Hence the ratio of the secondary terminal voltage to the primary impressed voltage is

$$\frac{v_1}{v_2} = \frac{N_1}{N_2} \tag{30}$$

That is, the two terminal voltages are in the same ratio as the numbers of turns in the two windings.

When an external circuit (*e.g.*, a group of lamps or an alternating-current motor) is connected to the terminals of the secondary winding, this potential difference $v_2$ will cause a current $i_2$ to flow in the closed circuit thus formed. When the secondary gives out electric energy to this external circuit, the current $i_2$ must be in the direction of the electromotive force $e_2$ induced in the secondary, and therefore in the *left-handed* screw direction with respect to the mutual flux $\phi$, whose rate of change produces this electromotive force. The secondary current therefore tends to decrease this mutual flux $\phi$. However, for the same primary impressed voltage $v_1$, the value of this mutual flux $\phi$ must be the same when there is a current in the secondary winding

as when there is no current in this winding, for $N_1 \dfrac{d\phi}{dt}$ must still be equal to the primary impressed voltage $v_1$, since, by hypothesis, the resistance and leakage inductance of each of the two windings are negligible.

Consequently, the primary current $i_1$ must be greater than the "no-load" primary current $i_{10}$ by such an amount that the total magnetomotive force acting on the path of the flux $\phi$, for a given primary impressed voltage $v_1$, remains unaltered. That is, the value of $(N_1 i_1 - N_2 i_2)$ for a given impressed voltage $v_1$ must be independent of the value of the secondary current $i_2$. Hence, on the assumption of negligible resistance and leakage inductance, the primary current $i_1$, when the secondary current is $i_2$, must be

$$i_1 = i_{10} + \left(\frac{N_2}{N_1}\right) i_2$$

where $i_{10}$ is the primary current when there is no secondary current. Comparing this expression with equation (19a), Article 125, it is evident that, on the assumption of negligible resistance and leakage inductance, the exciting current $i_e$ is equal to the no-load primary current, viz.,

$$i_e = i_{10} = i_1 - \left(\frac{N_2}{N_1}\right) i_2$$

Since the mutual flux is not altered by the presence of the secondary current, the secondary induced voltage $e_2$ will likewise remain unaltered. Moreover, since by hypothesis the resistance and leakage inductance of the secondary winding are negligible, the secondary terminal voltage $v_2$ will still be equal to $e_2$, and therefore to $\dfrac{N_2}{N_1} v_1$.

The electric power input to the primary winding when the primary current is $i_1$ is $p_1 = v_1 i_1$, and the corresponding electric power output of the secondary winding (to the circuit connected in series with it) is $p_2 = v_2 i_2$. But $i_1 = i_{10} + \left(\dfrac{N_2}{N_1}\right) i_2$ and $v_2 = \left(\dfrac{N_2}{N_1}\right) v_1$. Therefore

$$p_1 = v_1 i_1 = v_1 \left(\frac{N_2}{N_1}\right) i_2 + v_1 i_{10}$$

$$p_2 = v_2 i_2 = v_1 \left(\frac{N_2}{N_1}\right) i_2$$

Hence, neglecting the resistances and leakage inductances of the two windings, the power input to the primary winding for any given value of the primary voltage is equal to the power output of the secondary winding, plus an amount of power $v_1 i_{10}$ equal to the power input to the primary when there is no load on the secondary. This component $v_1 i_{10}$ of the primary power input is equal at each instant to the power lost in the core due to hysteresis and eddy currents, plus the rate at which magnetic energy is being stored in the magnetic circuit at this instant.

From the relations just developed it follows that the *secondary terminal voltage*, when the resistances and leakage fluxes are neglected, *is fixed by the primary impressed voltage*, and is equal to the primary impressed voltage multiplied by the ratio of the primary to the secondary turns. The *secondary current*, on the other hand, *is determined solely by the characteristics of the external circuit connected to the terminals of the secondary winding*, *i.e.*, by the resistance, inductance and electromotive force in this external circuit. Under normal operating conditions, the secondary current $i_2$ usually has such a value that the corresponding component $\left(\dfrac{N_2}{N_1}\right) i_2$ of the primary current is very much larger than the no-load primary current $i_{10}$. Hence, for normal load on a transformer, the power input to the primary is approximately equal to the power output of the secondary, viz., $v_1 i_1 = v_2 i_2$, and, since $\dfrac{v_1}{v_2} = \dfrac{N_1}{N_2}$ approximately, the ratio of the primary to the secondary current is approximately *inversely* as the ratio of the numbers of turns in these two windings, viz.,

$$\frac{i_1}{i_2} = \frac{N_2}{N_1} \qquad \text{approximately} \qquad (30a)$$

That winding of a transformer which has the larger number of turns, and therefore the higher terminal voltage, is usually referred to as the "high-tension" winding, and the winding with the fewer turns is called the "low-tension" winding. Either winding may be connected to the source of power, *i.e.*, either winding may be made the "primary" winding. When the low-tension winding is the primary, the transformer is called a "step-up" transformer, since the secondary terminal voltage is then greater than the primary impressed voltage. When the high-tension winding is made the primary, the transformer is called a "step-down" transformer. The ratio of the number of turns in

the high-tension winding to the number of turns in the low-tension winding is called the "turn ratio" of the transformer; this ratio is never less than unity.

In a practical transformer, the resistances and leakage inductances of the two windings are always of appreciable magnitude, and the current and voltage relations just deduced are therefore approximate only. The exact current and voltage relations may, however, always be expressed in terms of the induction coefficients of the two windings and their resistances. From equation (23) the total back electromotive induced in the primary is

$$e_1 = \frac{d\lambda_1}{dt} = \left(\frac{N_1}{N_2}\right) \frac{d}{dt}(Mi_e) + \frac{d}{dt}(L_{10}i_1) \tag{31}$$

and from equation (23a) the resultant electromotive force induced in the secondary winding, in the direction of the secondary current $i_2$, is

$$e_2 = -\frac{d\lambda_2}{dt} = \frac{d}{dt}(Mi_e) + \frac{d}{dt}(L_{20}i_2) \tag{31a}$$

where $i_e$ is the exciting current, referred to the primary, viz.,

$$i_e = i_1 - \left(\frac{N_2}{N_1}\right) i_2 \tag{31b}$$

In these expressions $M$ is the mutual inductance between the two windings and $L_{10}$ and $L_{20}$ are the leakage inductances of the two windings, as calculated from equations (19), (21) and (21a) respectively.

The term $\frac{d}{dt}(Mi_e)$ in equation (31a) represents the electromotive force induced in the secondary due to the mutual flux between this winding and the primary winding. Let $e'$ represent this electromotive force. On the assumption that the leakage inductances $L_{10}$ and $L_{20}$ are constants, as is usually the case, at least to a close degree of approximation, equations (31) and (31a) may be then written

$$e_1 = \left(\frac{N_1}{N_2}\right) e' + L_{10} \frac{di_1}{dt}$$

$$e_2 = e' - L_{20} \frac{di_2}{dt}.$$

Hence, calling $r_1$ and $r_2$ the resistances of the primary and secondary windings respectively, the primary terminal voltage $v_1$ ( $= e_1 + r_1 i_1$) and secondary terminal voltage $v_2$ ( $= e_2 - r_2 i_2$) are respectively

$$v_1 = \left(\frac{N_1}{N_2}\right) e' + r_1 i_1 + L_{10} \frac{di_1}{dt} \qquad (32)$$

$$v_2 = e' - r_2 i_2 - L_{20} \frac{di_2}{dt} \qquad (32a)$$

From these two equations and the relation that

$$e' = \frac{d}{dt}(Mi_e) \qquad (32b)$$

the primary current and secondary terminal voltage may be found for any given load (*e.g.*, lamps or motors) connected to the secondary terminals. The complete solution of these equations, however, leads to a more detailed discussion of the alternating-current transformer than can be given here.

**Problem 12.**—A transformer which has a turn ratio of 10:1 has 120 turns in its low-tension winding. The resistance of the high-tension winding is 15 ohms, and the resistance of the low-tension winding is 0.2 ohms. The mutual inductance between the two windings is 20 henries, the leakage inductance of the high-tension winding is 0.1 henries and the leakage inductance of the low-tension winding is 0.0008 henry. The transformer is used as a step-up transformer. At a certain instant the current in the primary winding is 158 amperes and the current in the secondary winding is 15 amperes. At this particular instant each current is increasing, the primary current at a rate of 53,000 amperes per second, and the secondary current at a rate of 5150 amperes per second. Assume the mutual and leakage inductances to be constant. Calculate, for the particular instant specified, the following:

(*a*) The exciting current referred to the primary. (*b*) The rate of change of this exciting current. (*c*) The induced voltage in the secondary *due to the mutual flux*. (*d*) The *total* induced voltage in the secondary. (*e*) The *total* induced voltage in the primary. (*f*) The secondary terminal voltage. (*g*) The primary impressed voltage. (*h*) The ratio of the secondary terminal voltage to the primary impressed voltage.

*Answer.*—(*a*) 8 amperes. (*b*) 1500 amperes per second. (*c*) 30,000 volts. (*d*) 29,485 volts. (*e*) 3042 volts. (*f*) 29,260 volts. (*g*) 3073 volts. (*h*) 9.52.

**128. Alternating-current Generator.**—An alternating-current generator, or "alternator," consists essentially of two windings, one stationary and the other mounted so that it can be rotated (by means of steam engine or other driving motor) about an axis perpendicular to its own axis. One of the windings, called the "field winding" is supplied with a direct (non-varying) current. The other winding, in which an alternating electromotive force is induced by the relative motion of the two windings, is called the "armature winding." Either the field winding or the arma-

ture winding may be the stationary one. In practically all modern alternators it is the field winding which is rotated.

The essential features of an alternator may best be understood by considering a simple, two-pole, stationary field type, as shown in Fig. 93. The field and armature cores are identical in construction with those of a direct-current generator (Article 83), except that the field cores are usually laminated. The field winding is identical with that of a shunt generator, *i.e.*, has a large number of turns of relatively small wire. The
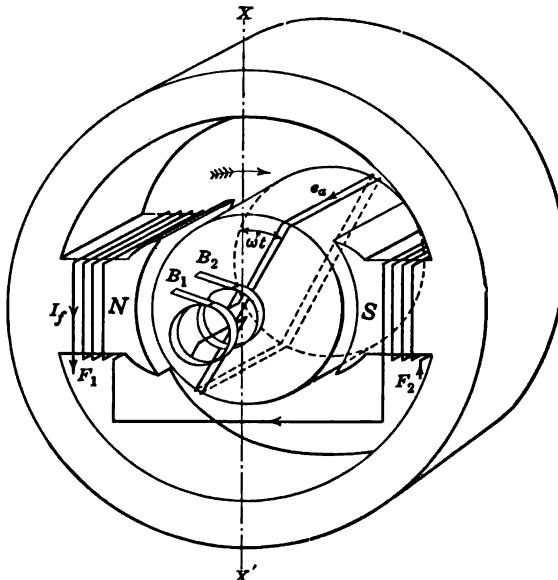


FIG. 93.—Two-pole alternator with stationary field.

armature winding is also similar to that of a direct-current generator, except that, in the single-circuit, or "single-phase," alternator it usually covers only a part of the armature core.

The fundamental difference in construction between an alternator and a direct-current generator is that an alternator has no commutator, but instead, the two ends of the armature winding are brought out to two closed rings, as shown in Fig. 93, called "slip-rings." Carbon or copper brushes $B_1$ and $B_2$, rubbing on these rings, serve to connect the armature winding to the external circuit to which electric energy is to be supplied.

The current for the field winding, which is a direct current, is usually supplied from a small direct-current generator (called

an "exciter"), which may be an entirely separate machine, or may have its armature mounted on the shaft of the alternator.

An inspection of Fig. 93 will show that as the armature rotates in the magnetic field due to the field current, the number of lines of force which link the coil formed by the armature winding varies continuously, being a maximum when the axis of the armature coil coincides with the axis of the field coils, and zero when the axis of the armature coil is perpendicular to the axis of the field coils. Also, as the armature rotates, the lines of force enter first one face, and then the other face, of the armature winding. Hence, during a complete revolution, the flux which links the armature winding in the right-handed screw direction with respect to the direction around the winding from the brush $B_1$ to the brush $B_2$, changes from a maximum positive value to an equal negative value and back again to its original positive value, repeating this cycle for each revolution.

This varying flux will therefore induce in the armature winding an electromotive force which will vary from a maximum value in one direction to an equal value in the opposite direction, and back again to its original value in the first direction, repeating this cycle of values for each revolution of the armature. Consequently, when the armature is driven at a constant speed of, say, $n$ revolutions per minute, an *alternating* electromotive force is established between the brushes $B_1$ and $B_2$, this electromotive force passing through $\frac{n}{60}$ complete cycles of values per second. This last relation is for a *two-pole* machine; when the alternator has $p$ poles, the "frequency," or number of times per second that the electromotive force passes through a complete cycle of values, is

$$f = \frac{np}{120} \qquad (33)$$

When an external circuit is connected to the brushes $B_1$ and $B_2$, this alternating electromotive force will establish in this circuit an alternating current, which will pass through $f$ complete cycles of values each second, *i.e.*, will have this same frequency $f$. The value of this current at any instant will depend upon (1) the value of the electromotive force induced in the armature at this instant by its rotation in the magnetic field, upon (2) the resistance and inductance of the armature, and upon (3) the

resistance, inductance and electromotive force (if any) in the external circuit.

The terminal voltage of the armature, *i.e.*, the voltage between the brushes $B_1$ and $B_2$, will also be an alternating voltage, and its value at each instant will, in turn, depend upon the current in the armature and the internal resistance and inductance of the armature.

Consider first the case when there is no current in the armature winding, *i.e.*, when there is no external circuit connected to the brushes $B_1$ and $B_2$. Under these conditions the only change in the flux linkages of the armature winding is that due to its rotation in the field due to the field current. These flux linkages will be a maximum when the axis of the armature winding coincides with the axis of the field winding, *i.e.*, when the angle $\omega t$ in Fig. 93 is zero. Let $\phi_m$ be the total flux through the armature when in this position. The field current $I_f$ required to establish this flux may be calculated in exactly the same manner as for a direct-current generator (see (Article 91).

Due to the fact that the armature winding is distributed over the surface of the armature core, all the armature turns $N_a$ will not, at any given instant, be linked by the same number of lines of force. Let $(k_a N_a)$ be the average number of armature turns linked by the lines of force which represent the flux $\phi_m$, when the axis of the armature winding coincides with the axis of the field winding, that is, when $\omega t = 0$. For a distributed armature winding the factor $k_a$ is always less than unity. It is called the "distribution factor" of the winding.

The total flux linkages of the armature winding when its axis coincides with the axis of the field winding, is then

$$\lambda_m = (k_a N_a)\phi_m \qquad (34)$$

As already explained, when the armature rotates, the linkages between its turns and the flux $\phi_m$ decrease from this value $\lambda_m$ to zero (corresponding to $\omega t = \dfrac{\pi}{2}$ in Fig. 93), then increase to $\lambda_m$ in the opposite direction with respect to the direction around the armature from $B_1$ to $B_2$ (corresponding to $\omega t = \pi$), then decrease to zero again (corresponding to $\omega t = \dfrac{3\pi}{2}$), and then increase to their original value in the original direction (corresponding to $\omega t = 2\pi$).

The exact relation between these flux linkages and time, as the armature rotates, will depend upon the distribution of the lines of force in the air-gap. By properly shaping the pole faces it is possible to make this distribution such that at any instant the linkages $\lambda_a$ between the flux $\phi_m$ and the armature turns may be represented, to a close degree of approximation, by a sinusoidal function of time, viz., by the relation

$$\lambda_a = \lambda_m \cos \omega t \tag{35}$$

In this expression $t$ is time in seconds measured from the instant at which the axis of the armature winding coincides with the axis of the field winding and, for a two-pole machine, $\omega$ is equal to $2\pi$ times the number of revolutions per second. The angle $\omega t$ is then the angle, in radians, at any instant between the axis of the armature winding and the axis of the field winding. In the general case of a machine which has $p$ poles.

$$\omega = 2\pi f \tag{36}$$

where $f$ is the frequency in cycles per second, as given by equation (33).

The electromotive force induced in the armature winding at any instant is then

$$e = \frac{d\lambda_a}{dt} = \omega \lambda_m \sin \omega t \tag{37}$$

Or, substituting for $\omega$ and $\lambda_m$ their values, as given by equations (36) and (34), the electromotive force induced at any instant in the armature winding of an alternator, when there is no current in this winding, may be written

$$e = E_m \sin (2\pi f t) \tag{37a}$$

where, when $\phi_m$ is in maxwells,

$$E_m = 2\pi f \ (k_a N_a)\phi_m \times 10^{-8} \text{ volts} \tag{37b}$$

The electromotive force $E_m$ given by equation (37b) is the maximum value reached, during each cycle, by the electromotive force induced in the armature winding. As already noted, and as shown by equation (37a), this induced electromotive force varies during each cycle from this maximum value $+E_m$ to $-E_m$ and back again to $+E_m$. A negative value of induced electromotive force $e$ merely signifies that its direction, instead of being from the brush $B_1$ to the brush $B_2$, is from the brush $B_2$ to the brush $B_1$.

When the alternator supplies current to an external circuit, this current, which also alternates between equal values in opposite directions, flows through the armature winding and therefore modifies the flux which links the armature turns. The effects of the armature current are too complex to be considered in detail here. It should be noted, however, that these effects can be expressed quantitatively in terms of the self- and mutual inductances of the field and armature windings, in much the same way that the effects produced by the currents in the two windings of a transformer are expressed.

In the case of an alternator, however, the variation in the values of the induction coefficients with time, due to the relative motion of the two windings, must be taken into account. A brief consideration of these variations will be instructive at this point.

Were the permeability of the iron constant, the self-inductance $L_f$ of the field winding would be constant, except for the pulsations in the reluctance of the path of the flux, due to the motion past the pole faces of the slots in the armature core. Actually, however, the permeability of the iron depends upon the *resultant* flux density established in it, which in turn depends upon both the field current and the armature current. Hence, due to the variations in the armature current with time, the self-inductance of the field winding, even for a constant field current, is only approximately constant.

The self-inductance $L_a$ of the armature winding is likewise a function of time, and to an even more marked extent, for the reluctance of the path of the flux which is produced by a current in this winding is greater when its axis is at right angles to the axis of the field $\left(\omega t = \dfrac{\pi}{2}\right)$ than when the axes of the two windings are parallel ($\omega t = 0$). This is due to the fact that when the armature winding is in the first position, the path of the flux produced by a current in this winding will pass through the air space between the poles, whereas when this winding is in the second position, this flux will pass through the field cores and yokes. The self-inductance $L_a$ will therefore have its maximum value when $\omega t = 0$ and its minimum value when $\omega t = \dfrac{\pi}{2}$.

The mutual inductance $M$, as may be readily seen from Fig. 93,

will decrease from a maximum value, say $M_m$, when the axes of the armature and field windings coincide ($\omega t = 0$), to zero when these axes are at right angles $\left(\omega t = \dfrac{\pi}{2}\right)$. Also, as the armature rotates through the next quarter revolution, $M$ will increase from 0 to $M_m$, will then decrease from $M_m$ to zero for the third quarter revolution, and increase from 0 to $M_m$ for the last quarter.

As already noted, during the first and fourth quarter revolutions the flux linkages of the armature winding, due to the mutual flux, are in the right-handed screw direction with respect to the direction around this winding from the brush $B_1$ to the brush $B_2$, whereas during the second and third quarter revolutions these flux linkages are in the *opposite* direction. Hence, since the positive sense around the armature winding is taken as the direction through the armature from $B_1$ to $B_2$, the flux linkages of the armature current in the right-handed screw direction with respect to this positive sense changes from $+M_m I_f$, when $\omega t = 0$, to $-M_m I_f$, when $\omega t = \pi$, and back again to $+M_m I_f$, when $\omega t = 2\pi$. This fact may be conveniently taken into account by considering the mutual inductance $M$ as passing through a complete cycle of values during each revolution, from a positive maximum $+M_m$ to a negative maximum $-M_m$ and back again to $+M_m$.

Let $i_f$ and $i_a$ be the currents in the field and armature windings at any instant, in the directions indicated by the arrows in Fig. 93. The total flux linkages of these two windings at this instant, in the right-handed screw direction with respect to the currents, are then respectively

$$\lambda_f = L_f i_f + M i_a \tag{38}$$
$$\lambda_a = L_a i_a + M i_f \tag{38a}$$

The electromotive forces induced in these two windings at this instant, in the directions of the currents, are then $e_f = -\dfrac{d\lambda_f}{dt}$ and $e_a = -\dfrac{d\lambda_a}{dt}$.

Let $r_f$ and $r_a$ be the resistances of the field and armature windings respectively. Let $v_f$ be the voltage across the terminals of the field winding, and let $v_a$ be the terminal voltage of the armature at the instant under consideration, *i.e.*, the net rise of potential from the brush $B_1$ to the brush $B_2$. Then at this instant $v_f = r_f i_f - e_f$ and $v_a = e_a - r i_a$, whence

$$v_f = r_f i_f + \frac{d\lambda_f}{dt} \tag{39}$$

$$v_a = -\frac{d\lambda_a}{dt} - r_a i_a \tag{39a}$$

where $\lambda_f$ and $\lambda_a$ are given by equations (38) and (38a).

These are the general equations which must be used in order to take into account all the components of the electromotive forces induced in the two windings. However, as a first approximation, the two self-inductances $L_f$ and $L_a$ may be assumed constant, and the mutual inductance may be assumed to have at any instant $t$ the value $M = M_m \cos \omega t$. Also, as a first approximation, the alternating electromotive force induced in the field winding may be neglected, and the field current may be assumed constant and equal to

$$I_f = \frac{v_f}{r_f} \tag{39b}$$

On these assumptions, the armature terminal voltage $v_a$ becomes

$$v_a = \omega M_m I_f \sin(\omega t) - \left(r_a i_a + L_a \frac{di_a}{dt}\right) \tag{40}$$

The electromotive force represented by the term $(\omega M_m I_f \sin \omega t)$ is commonly referred to as the "electromotive force of rotation." or as the "generated" electromotive force, since it is due to the relative motion of the armature and field. Comparing equations (40) and (37a), and noting that $v_a = e_a$ when there is no armature current, it is evident that the maximum value of the "generated" electromotive force, namely $\omega M_m I_f$, is equal to the electromotive force $E_m$, given by equation (37b). This, of course, is on the assumption that $M_m$ is independent of the value of the armature current, which is only approximately true. Equation (40) may then be written

$$v_a = E_m \sin(\omega t) - \left(r_a i_a + L_a \frac{di_a}{dt}\right) \tag{40a}$$

The term $r_a i_a$ is the resistance drop in the armature winding, and the term $L_a \frac{di_a}{dt}$ is the back electromotive force induced in the armature due to its total self-inductance. Equation (40a) therefore merely states that the generated voltage varies sinusoidally with time, and that the terminal voltage at any instant is less than the generated voltage at this instant by an amount equal

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to (1) the resistance drop in the armature plus (2) the back electromotive force of self-induction.

**Problem 13.**—The total flux entering the armature of a two-pole single-phase alternator is $35 \times 10^6$ maxwells, and the field current required to produce this flux is 150 amperes. The armature is driven at a constant speed of 3600 revolutions per minute. The armature winding has 40 turns, and the distribution factor for this winding is 0.6. The self-inductance of the armature winding, assumed constant, is 0.5 millihenry, and the resistance of the armature winding is 0.04 ohm. Assume the flux through the armature winding to vary sinusoidally with time.

(a) What is the maximum value of the electromotive force induced in the armature? (b) If, when the axis of the armature winding makes an angle of 45 degrees with the axis of the field winding, the armature current is 2500 amperes, and is increasing at the rate of 500,000 amperes per second, what is the armature terminal voltage at this instant? (c) What is the maximum value of the mutual inductance between the armature and field windings?

*Answer.*—(a) 3167 volts. (b) 1890 volts. (c) 56 millihenries.

# XI

## MECHANICAL FORCES IN MAGNETIC FIELDS

**129. General.**—Experiment shows that every current-carrying conductor and every magnetized substance in a magnetic field is in general acted upon by a mechanical force due to the agent which produces this field. More specifically,

(a) Between every two current-carrying conductors there exists a mechanical force whose value depends, among other things, upon the product of the currents in the two conductors (see Article 123).

(b) Between every current-carrying conductor and every magnetic pole there exists a mechanical force whose value depends, among other things, upon the product of the current in the conductor and the strength of the pole.

(c) Between every two magnetic poles there exists a mechanical force whose value depends, among other things, upon the product of the strength of the two poles.

These mechanical forces may be briefly described as forces exerted on the conductor or magnet by the magnetic field in which it is located. It should always be kept in mind, however, that the magnetic field is merely the medium through which one agent (a current-carrying conductor or magnet) exerts a force on the other agent (a second current-carrying conductor or magnet), and that whenever a force is thus exerted by one agent or another, the second agent exerts an equal and opposite force on the first. In other words, "action and reaction are always equal and opposite."

The value of the mechanical force exerted by a magnetic field on an electric circuit may always be deduced from the fundamental principle stated in Article 123, namely, that **the mechanical work done by this force, when the circuit is moved, is equal to the amount of electric energy transferred to this circuit as a result of this motion.**

From this general principle it follows, as shown in Article 123, that the component $f_s$ of the mechanical force exerted on an

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electric circuit in any direction is equal to the product of the current $i$ in this circuit multiplied by the increase in the number of flux linkages of this circuit per unit *linear* displacement of the circuit in this direction. That is

$$f_x = i\frac{d\lambda}{dx} \tag{1}$$

where $d\lambda$ is the increase in the flux linkages of the circuit, in the right-handed screw direction with respect to the current $i$, corresponding to a linear displacement $dx$ of the circuit in the given direction.

When the circuit consists of but a single turn of wire, the flux linkage $\lambda$ are equal to the flux $\phi$ which threads this turn. In this case this relation may be written

$$f_x = i\frac{d\phi}{dx} \tag{1a}$$

This formula is applicable not only to a complete circuit, but to any length of the wire in the circuit, provided $d\phi$ is taken as the increase in the flux which links this particular length of wire, or to the number of lines of force cut by this length of wire, when it is displaced the distance $dx$.

In exactly the same manner as equation (1) was deduced (see Article 123) it may be shown that the torque, or mechanical couple, $T$, tending to turn an electric circuit about any given axis, is equal to the product of the current $i$ in this circuit multiplied by the increase in the number of flux linkages of this circuit per unit *angular* displacement of the circuit about this axis. That is[1]

$$T = i\frac{d\lambda}{d\theta} \tag{2}$$

where $d\lambda$ is the increase in the flux linkages of the circuit, in the right-handed screw direction with respect to the current $i$, corresponding to an angular displacement $d\theta$ of the circuit about the given axis.

[1] Equation (2) may be deduced as follows: (a) the mechanical work corresponding to a torque $T$ and an angular displacement $d\theta$ is $Td\theta$; (b) the corresponding induced electromotive force is $e = \frac{d\lambda}{dt}$, where $dt$ is the time required for the displacement $d\theta$; (c) the electric energy which must be supplied to the circuit to keep the current constant at the value $i$ is $eidt = id\lambda$; (d) equating this to the mechanical work $Td\theta$ gives for the torque $T$ the value stated in equation (2).

The *direction* of the force or torque produced by a magnetic field on a conductor which carries an electric current may always be determined from the general rule that every part of an electric circuit *tends to move into such a position as will increase the number of lines of force which link this circuit in the right-handed screw direction with respect to the current in it.*

For example, two parallel conductors which carry currents in opposite directions (as in a transmission line) always **repel** each other, since the further apart the conductors get the greater will be the number of lines of force between them. Again, two coils with their faces parallel and their axes coinciding will attract each other when the currents in them are in the same direction, but will repel each other when the currents are in opposite directions.

From the fundamental relations above stated in regard to the
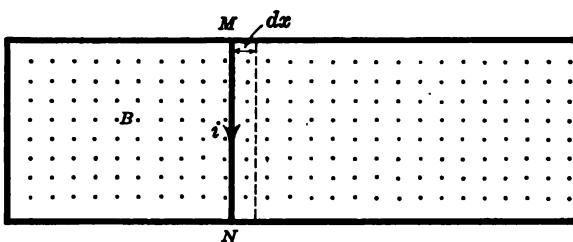


FIG. 94.

magnitude and direction of the mechanical forces in a magnetic field, numerous special formulas applicable to particular cases which arise in practice may be deduced. The more important of these special formulas will now be developed.

**130. Force Exerted on a Straight Wire by a Uniform Magnetic Field.**—Consider a straight wire $MN$ so placed in a magnetic field that it is perpendicular to the lines of force, as indicated in Fig. 94. The dots represent the intersections of the lines of force with the plane of the paper. The direction of these lines of force is assumed to be away from the eye of the reader. This wire may be thought of as part of a closed circuit, as indicated in the figure.

Let $l$ be the length of the wire $MN$ in centimeters, $i$ the current in it in abamperes, and $B$ the flux density of the magnetic field at the wire. Were the wire moved parrallel to itself, perpendicular to the lines of force, a distance $dx$ to the right, the increase in the

flux through the loop *CMND*, in the right-handed screw direction with respect to the current $i$, would be $d\phi = Bldx$. Whence, from equation (1a) the force exerted on the wire by the magnetic field is

$$f = Bli \qquad \text{dynes} \qquad \cdot \qquad (3)$$

A motion of the wire parallel to the lines of force would result in no change in the number of lines of force through the loop of which it forms a part; hence this is the resultant force acting on the wire.

Note that the direction of this force may be specified, without reference to the loop of which the wire forms a part, as follows. Point the *forefinger* of the *left hand* in the direction of the *lines of force*, the *middle finger* in the direction of the *current*, and hold the thumb perpendicular to these two fingers; the *thumb* will then point in the direction in which the wire tends to *move*. This rule is usually referred to as the "left-hand rule." Compare with the "right-hand rule" for the direction of the electromotive force induced in a wire when it is moved across a magnetic field (Article 82). Also compare the above formula for the mechanical force acting on the wire, with the formula $e = Blv$ for the electromotive force induced in a wire when moved with a velocity of $v$ centimeters per second in a direction perpendicular to its length and to the lines of force (see Problem 3, Article 81).

It should be carefully noted that equation (3) gives the force acting on the wire only when the wire and the lines of force are *mutually perpendicular*. When the lines of force and the wire make any angle $\theta$ with each other the resultant force on the wire is

$$f = Bli \sin \theta \qquad \text{dynes} \qquad (3a)$$

The direction of this force is always perpendicular to the plane determined by the wire and the direction of the lines of force.

**Problem 1.**—In Fig. 95 are shown the two poles of an electromagnet. The total flux in the air-gap is 400,000 maxwells. The side *AB* of the air-gap is 4 inches and the side *AC* is 2 inches. A single loop of wire carrying a current of 200 amperes is held in the air-gap in the position shown in the figure, the current being in the direction indicated by the arrow.

(a) Assuming the lines of force in the air-gap to be straight and uniformly distributed, what is the force in pounds required to hold the wire in the position shown? (b) What is the direction of this force? (c) How much mechanical work would be done were the wire allowed to move across the air-gap from the lower to the upper edge of the gap? (d) Assuming the lines

of force to be confined entirely to the air-gap, by how much would the force on the wire be changed were the length of the horizontal portion of the wire doubled?

*Answer.*—(a) 3.54 pounds. (b) Upward, from $C$ to $A$. (c) 0.59 foot-pound. (d) Not at all, provided the opposite ends of the horizontal portion of the wire are kept on opposite sides of the magnet, as in the figure.

**Problem 2.**—(a) Prove that the force of repulsion between the two wires of a transmission line, or feeder, carrying a current $i$ is

$$f = 0.54 \times 10^{-6} \frac{li^2}{D} \qquad \text{pounds} \qquad (5)$$

where $l$ is the length of each wire in feet, $i$ the current in amperes, and $D$ the distance between the centers of the two wires in inches. (b) The leads from a 5000-kilowatt generator are cables each 1,000,000 circular mils in cross-section. These cables are held in clamps attached to a wall. The
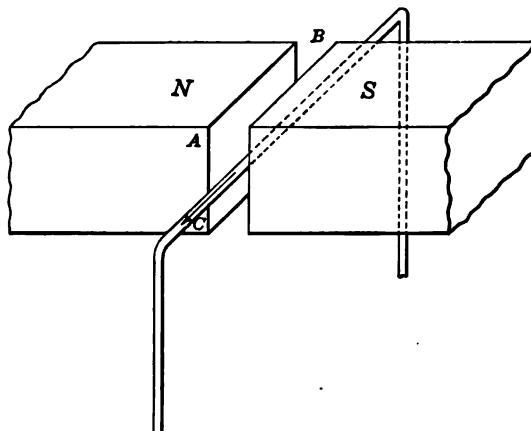


Fɪɢ. 95.

clamps are 6 feet apart, and the distance between the centers of the two cables is 6 inches. The feeder is accidentally short-circuited, and during the short circuit the current reaches a value of 20,000 amperes. What is the force in pounds exerted on each clamp tending to shear it off from the wall?

*Answer.*—(b) 216 pounds. (Nᴏᴛᴇ.—Cases have occurred in practice where this force of repulsion, due to a sudden short circuit, has been sufficient actually to tear the cables from their supports.)

**131. Force of Attraction or Repulsion Between Two Coils.**— An expression for the force exerted by one coil on another has already been deduced in Article 123, viz., the force exerted by one coil on another in any direction $dx$ (see Fig. 88) is

$$f_x = i_1 i_2 \frac{dM}{dx} \qquad \text{dynes} \qquad (6)$$

where $i_1$ and $i_2$ are the currents in the two coils, and $dM$ is the increase in their mutual inductance when one coil is given a linear displacement of $dx$ centimeters with respect to the other. In this formula $i_1$ and $i_2$ are in abamperes and the mutual inductance $M$ in abhenries.

An important application of this formula is the determination of the mutual force between the movable and fixed coils of an absolute current balance (see Article 33). This requires first a determination of the mutual inductance between the two coils, and then the differentiation of this mutual inductance with respect to the distance between the coils. This deduction is too complicated to be given here, but it should be noted that the calculation may be effected with a high degree of precision. It is on this account that the current balance may be used as a means for the accurate comparison of the practical and the c.g.s. electromagnetic units of current, viz., the ampere and the abampere.

From equation (6) it is evident that when the same current is caused to pass through each of two coils in series the force exerted by one coil on the other varies as the *square* of this current. This is a fundamental characteristic of a current balance.

**Problem 3.**—Referring to Fig. 18, p. 59, the mutual inductance between the movable coil at each end of the balance and each stationary coil is 0.001 millihenry when the movable coil is midway between the two fixed coils. When the movable coil is displaced a distance of 0.1 millimeter vertically downward, its mutual inductance with respect to the lower fixed coil increases by 0.000006 millihenry, and its mutual inductance with respect to the upper fixed coil decreases by the same amount. The distance from the knife edge to the center of each movable coil is 20 centimeters. What weight, hung from the beam at a distance of 10 centimeters from the knife edge, will be required to maintain the beam in its equilibrium position when a current of 50 amperes is established through the balance?

*Answer.*—122.4 grams or 4.32 ounces.

**132. Torque Exerted on a Coil by a Magnetic Field.**—As noted in Article 129, when a coil in a magnetic field carries a current $i$, it is in general acted upon by a torque, or couple, as well as by a direct pull or push. As shown in that article, the value of the torque tending to turn the coil about any axis is

$$T = i \frac{d\lambda}{d\theta}$$

where $d\lambda$ is the increase of the flux linkages of the coil, in the

right-handed screw direction with respect to the current $i$, when the coil turns through an angle of $d\theta$ radians about this axis. In this expression, when $i$ is in abamperes, and $\lambda$ in maxwells, the torque is in c.g.s units, viz., dyne-centimeters.

Consider a coil which is so mounted that it is free to turn about an axis which coincides with one of its *diameters* (or, when the coil is not circular, with a line which passes through its center parallel to its face, and which divides the coil into two symmetrical parts). This is the way in which the movable coil of a galvanometer, ammeter, or electrodynamometer is mounted.
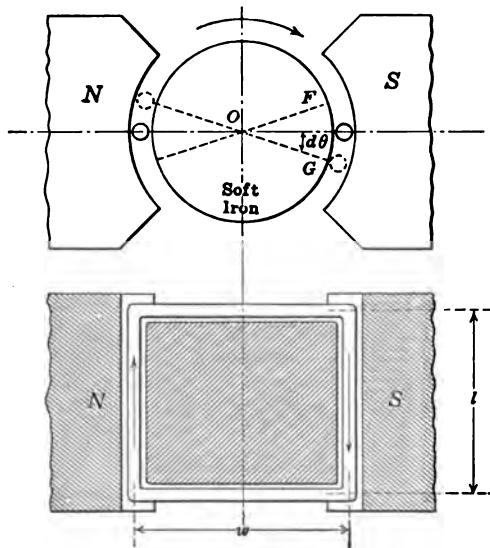


FIG. 96.

The coils formed by the conductors on the armature of a two-pole dynamo are another illustration of this arrangement.

Consider first the case when the axis of the coil is in such a direction that it is linked by none of the lines of force due to the agents (other than the current in the coil itself) which produce the magnetic field in which it is located (see Fig. 96). This is the normal, or "zero" position for the moving coil of a galvanometer. Let $i$ be the current in the coil in abamperes, and let $B$ be the average value of the flux density at the points occupied by the wire which forms the sides of the coil which are parallel to the axis of rotation. Let $l$ be the mean length of the coil parallel to this axis, and let $w$ be its mean width perpendicular to this axis.

When the coil is displaced from its "zero" position by an angle $d\theta$, as indicated in the figure, the number of lines of force which thread it will be

$$d\phi = B(l \times \overline{FG})$$

where $\overline{FG}$ is the small arc indicated in the figure. But $\overline{FG} = 2 \times \overline{OF} \times d\theta$, and $\overline{OF} = \dfrac{w}{2}$. Consequently, $\overline{FG} = wd\theta$, and therefore $d\phi = B(lw)\,d\theta$. Let $N$ be the number of turns in the coil, and let these turns be so close together that the coil may be considered as a concentrated winding. Then the increase in the flux linkages of the coil when it turns through the angle $d\theta$ is

$$d\lambda = NB(lw)d\theta$$

Hence, from equation (2), when a current $i$ is established in the coil, the magnetic field in which it is located produces on it a torque of

$$T = (Nwl)\ Bi \qquad \text{dyne-centimeters} \qquad (7)$$

where all quantities are in c.g.s. electromagnetic units. This then is the torque which, in the case of a galvanometer, ammeter, or electrodynamometer, tends to deflect the movable coil from its zero position.

This formula can also be deduced directly from equation (3), noting that the force on each side of the coil is $f = N(Bli)$, and, since the forces on the two sides of the coil are in opposite directions, the couple formed by these two forces is $T = fw = wN(Bli)$.

When the magnetic field in which the coil rotates is produced by a permanent magnet, as in a galvanometer or direct-current ammeter (see Figs. 15 and 16), the flux density $B$ will be independent of the strength of the current $i$, and the torque exerted on the coil when a current is sent through it will therefore be directly proportional to the strength of this current.

By mounting inside the coil a fixed, soft iron core, as indicated in Fig. 96, the flux density in the air-gap may be made practically constant throughout a relatively large part of the air-gap, so that even for a relatively large displacement of the coil from its zero position (45 degrees or more in the case of an ammeter), the torque will still be proportional to the current. Hence, when the motion of the coil is opposed by a torsional force (due

to a fiber or spiral spring) which is proportional to its deflection, the amount by which the coil is deflected when a current is sent through it will be directly proportional to the strength of this current.

In an electrodynamometer the flux density $B$ at the wires of the movable coil is directly proportional to the current in the stationary coil. Consequently, when the instrument is used as an ammeter and the same current is sent through both coils, the torque exerted on the movable coil is proportional to the *square* of the current. This accounts for the fact that the scale of an electrodynamometer type of ammeter is not uniform. For example, when the division corresponding to 5 amperes, say, is 1 inch from the zero, the division corresponding to twice this current (10 amperes) is at a distance of 4 inches from the zero, or approximately this amount, depending on the design of the instrument. The electrodynamometer type of ammeter, however, has the advantage that it can be used to measure either direct or alternating currents.

Problem 4.—A certain millivoltmeter at full-scale deflection carries a current of 0.025 ampere. The movable coil is made of 200 turns of fine wire, wound in the form of a rectangle 1 inch by 1.5 inches. The flux density in the air-gap in which this coil moves is 800 lines per square centimeter. What is the torque at full-scale reading exerted on the movable coil by the permanent magnet? Give answer in gram-millimeters, *i.e.*, the force in grams which, acting at a radius of 1 millimeter, will balance this torque.

*Answer.*—39.5 gram-millimeters. (NOTE.—This is equivalent to a force of about ½ ounce acting at a radius of 0.1 inch.)

### 133. Average Torque Exerted by a Magnetic Field on a Rotating Coil.—In Fig. 97, is shown diagrammatically the field coils and armature conductors and the field and armature cores of a 2-pole direct-current dynamo. By means of the commutator the current in all those armature conductors which are under one pole is maintained in the same direction, and the current in all those armature conductors under the other pole is maintained in the opposite direction, as indicated by the plus and minus signs. Let $Z$ be the total number of armature conductors, let $\phi$ be the flux per pole which enters or leaves the armature core, and let $i_1'$ be the current in each armature conductor; this is *one-half* the current which is supplied to (or supplied by) the armature (see Article 83).

Consider two armature conductors in two diametrically opposite slots, for example $A_1$ and $A_2$. These conductors may be considered as forming a coil of one turn. When the two slots $A_1$ and $A_2$ are in the position shown in the figure, practically all the lines of force which represent the flux $\phi$ will thread this coil in the *left-handed* screw direction with respect to the current in it. When the armature has rotated through such an angle that the slot $A_1$ occupies the position $A'_1$ and the slot $A_2$ the position $A'_2$ (substantially 180°, or $\pi$ radians), the coil will again be threaded by substantially all the lines of force $\phi$, but in the *right-handed* screw direction with respect to the current in it.

Hence, during a half revolution, *i.e.*, during an angular displacement of $\pi$ radians, the flux through the coil increases by an
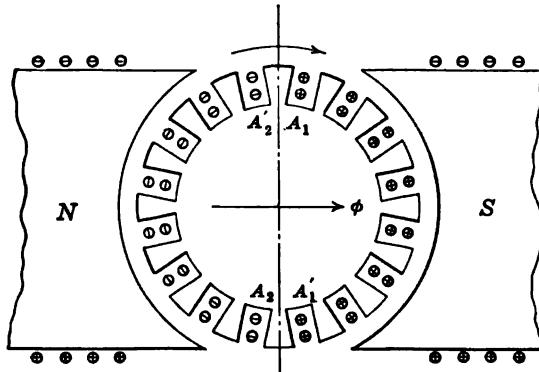


Fig. 97.

amount $2\phi$. Throughout this half revolution the current in the coil is in the same direction. Hence the average torque exerted on the armature by the field magnets, due to the current in this particular coil, is

$$T_a = i_1 \frac{2\phi}{\pi} = \frac{2}{\pi}\phi i_1'$$

When there are a large number of armature conductors, located in uniformly spaced slots as shown in the figure, the torque exerted on the armature at any instant due to the currents in all the $\frac{Z}{2}$ *pairs* of conductors must have this same value. Hence the total torque exerted on the armature at any instant by the field magnets is

$$T = \frac{Z}{\pi} \phi i_1' \tag{8}$$

In this formula all quantities are in c.g.s. electromagnetic units.

In exactly the same manner it may be shown that when the dynamo has $p$ poles instead of 2, and $a$ parallel paths between the brush sets instead of 2 as above assumed, the average torque exerted on the armature by the field magnets is

$$T_a = \frac{pZ}{2\pi a} \phi i \qquad \text{dyne-centimeters} \tag{8a}$$

where $\phi$ is the flux per pole and $i$ is the *total* current supplied to (or supplied by) the armature. In this formula all quantities are in c.g.s. electromagnetic units. When $\phi$ is expressed in maxwells and $i$ in amperes, the torque in pound-feet is

$$T_a = 0.1174 \times 10^{-8} \left(\frac{pZ}{a}\right) \phi i \qquad \text{pound-feet} \tag{8b}$$

That is, *the average torque exerted on the rotating member of a dynamo is always proportional to the product of the resultant flux per pole by the current in the winding on this member.* Note particularly that the torque developed by a motor for (1) a given armature current and (2) a given resultant flux $\phi$ per pole, is *independent of the speed* at which the armature rotates.

It should be noted that equation (8b) gives the gross torque developed by the armature of a motor. The net torque available at the pulley is this gross torque less that required to overcome the friction of the bearings and windage and the opposing torque due to the eddy-current and hysteresis loss in the magnetic circuit of the machine.

Problem 5.—The following data are for a certain 50-horsepower, 125-volt, shunt motor.

> Number of poles, 6.
> Number of conductors, 218.
> Number of parallel paths, 2.
> Resultant flux per pole at full load, $1.2 \times 10^6$ maxwells.
> Armature resistance (hot), 0.015 ohm.
> Friction, windage and core-loss at full speed, 3 horsepower.
> Power lost in field winding, 0.75 kilowatt.

(a) What is the gross torque developed by this motor when the armature current is 300 amperes. (b) What is the back electromotive force developed in the armature, when the current is 300 amperes and the impressed voltage 125? (c) What is the corresponding speed of the motor (see equation (3) of Article 84). (d) What is torque required to overcome friction, windage

and core-loss? (*e*) What is the net torque available at the shaft? (*f*) What is the horsepower output of the motor at the pulley when the current is 300 amperes? (*g*) What is the efficiency of the motor at this load?

*Answer.*—(*a*) 276 pound-feet. (*b*) 120.5 volts. (*c*) 921 revolutions per minute. (*d*) 17.1 pound-feet. (*e*) 259 pound-feet. (*f*) 45.4 horsepower. (*g*) 88.5 per cent.

**134. Forces on Magnetic Poles in a Non-magnetic Medium.—** As shown in Article 130, when a wire which has a length of *l* centimeters and which carries a current of *i* abamperes makes with the lines of force which represent a magnetic field the angle *θ*, the agent which produces this field exerts on the wire a mechanical force

$$f = Bli \sin \theta \qquad \text{dynes}$$

where *B* is the flux density produced at *P* by this agent. The direction of this mechanical force is perpendicular to the length
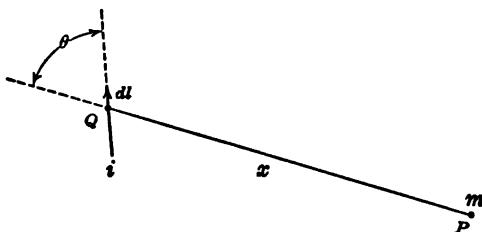


Fig. 98.

*l* and the lines of force which represent the magnetic field, in the sense determined by the "left-hand" rule (see Article 130).

Referring to Fig. 98, imagine at any point *P* a positive (or north) magnetic pole of strength *m*, concentrated in a *point* in a non-magnetic medium. Such a condition can never be actually realized, but is closely approximated by placing at *P* one end of a long slim magnet (see Article 104), surrounded by air or other non-magnetic medium. A magnetic pole which is confined to a surface so small that it may be considered as concentrated in a point is called a "point-pole."

Imagine at any other point *Q*, at a distance *x* from *P*, an elementary length *dl* of wire carrying a current of *i* abamperes, and let this elementary length make with the line $\overline{PQ}$ the angle *θ*. The magnetizing force at *Q* due to magnetic pole at *P* is $H_m = \dfrac{m}{x^2}$ (see Article 104), and, since by hypothesis the surrounding medium is non-magnetic, this is also equal to the flux density at

$Q$ due to the magnetic pole at $P$. Hence the mechanical force exerted on the elementary length $dl$ of wire is

$$df = \frac{m}{x^2} i \, dl \, (\sin \theta)$$

From the fundamental relations stated in Article 104, the expression

$$\frac{i(dl \sin \theta)}{x^2}$$

is equal to the magnetizing force $dH_c$ at $P$ due to the current $i$ in the elementary length $dl$. Whence, the mechanical force exerted on elementary length of wire at $Q$ by the pole at $P$ may also be written

$$df = m.dH_c \qquad \text{dynes} \qquad (9)$$

Moreover, since action and reaction are always equal and opposite, the mechanical force exerted on the magnetic pole $m$ by the current in the elementary length $dl$ is likewise equal to $m.dH_c$. Note also that this force is in the direction of the line of force at $P$ due to the current in $dl$. That is, the mechanical force exerted on the given north pole by the magnetic field due to the current is *in the direction of this field*, whereas the mechanical force exerted on a current-carrying conductor by a magnetic field is always perpendicular to the direction of this field and the direction of the current in the conductor. Were the given pole a south pole, the mechanical force exerted on it would be in the opposite direction (compare Article 75).

The resultant mechanical force exerted on the given magnetic pole by the currents in all the elementary lengths which make up all the electric circuits in the field is the vector sum of the forces as calculated from equation (9) for all these elements, *i.e.*, is equal to the strength of this pole multiplied by the resultant magnetizing force $H$ due to all these currents, or

$$f = mH \qquad \text{dynes} \qquad (9a)$$

On the hypothesis that a magnet is merely an aggregation of electric circuits of molecular dimensions (see Article 114), it would be expected that the mechanical force exerted by a magnet on the pole of another magnet would likewise be equal to the strength of the latter pole multiplied by the resultant magnetizing force at this pole due to the poles of the given magnet. Experiment is entirely in accord with this conclusion. Hence

equation (9a) is a perfectly general expression for the mechanical force exerted on a *point-pole* of strength $m$, provided this pole is in *air*, and $H$ is taken as the resultant magnetizing force at the point at which $m$ is located, due to all the electric currents in the field and all the magnetic poles in the field *other than* this particular point-pole.

From equation (9a) it is evident that the force in dynes exerted on a point-pole of unit strength is numerically equal to the magnetizing force at this point, due to all the agents in the field other than this particular point-pole, provided the point in question is in a non-magnetic medium. Hence the common definition of magnetizing force at any point as the force in dynes which would be exerted on a unit point-pole at this point. This definition, however, is not applicable to points within a magnetic substance. The various attempts which have been made to modify this definition to make it applicable to such points has been the source of much confusion. In the author's opinion, the definition of magnetizing force given in Article 90 is decidedly preferable to any definition based on the phenomenon of the mechanical force exerted on a magnetic pole.

The application of equation (9a) to the special case of two point-poles of strengths $m$ and $m'$ at a distance $x$ apart gives for the mechanical force between them the expression

$$f = \frac{mm'}{x^2} \qquad \text{dynes} \qquad (10)$$

This follows immediately from the fact that the pole $m'$ produces at a distance $x$ away a magnetizing force equal to $\dfrac{m'}{x^2}$ (see Article 104). Hence $m'$ exerts on $m$ a force of $\dfrac{mm'}{x^2}$ dynes, and $m$ exerts on $m'$ an equal and opposite force. When the two poles are of like sign the force is a repulsion, and when they are of unlike sign, the force is an attraction, *i.e.*, like poles repel each other and unlike poles attract each other.

Equations (9a) and (10) are of very little practical importance, for the reason that they apply only to poles which are concentrated in points, which condition is usually not even approximately realized.

Of course, if the distribution of the magnetic poles over the surfaces of the various magnets is known, it is possible to apply

these relations to each pair of points in these surfaces, and take the vector sum of the forces due to these point-poles. However, the distribution of the poles over the surface of a magnet can, as a rule, be calculated only to a rough degree of approximation.

**Problem 6.**—As a first approximation, the poles of a long cylindrical bar magnet may be assumed to be uniformly distributed over the two end surfaces of the magnet. Each end of the magnet may therefore be considered as a circular disc each unit area of which has the same pole strength. Let the numerical value of this pole strength per unit area be $\sigma$. Let $r$ be the radius of the magnet and let $l$ be its length. Let $P$ be a point on the axis of the magnet at a distance $a$ from its north pole. Let $\theta = \tan^{-1}\left(\dfrac{r}{a}\right)$ and let $\theta' = \tan^{-1}\left(\dfrac{r}{a+l}\right)$. All quantities are expressed in c.g.s. electromagnetic units.

(a) Prove that the magnetizing force at $P$ due to the north pole of the magnet is $2\pi\sigma\,(1 - \cos\,\theta)$ and that the magnetizing force at $P$ due to the south pole of the magnet is $2\pi\sigma\,(1 - \cos\,\theta')$ To prove this, find first the magnetizing force at $P$ due to a ring of radius $x$ and width $dx$, located in the north-pole end of the magnet, and integrate between the limits $x = 0$ and $x = r$; and similarly for the south pole (see also Problem 14, Article 107).

(b) Prove that when the point $P$ is outside the magnet the resultant magnetizing force at this point is

$$H_o = 2\pi\sigma\,(\cos\,\theta' - \cos\,\theta) \tag{11}$$

and that when $P$ is inside the magnet the resultant magnetizing force is

$$H_i = 2\pi\sigma\,[2 - (\cos\,\theta' + \cos\,\theta)] \tag{11a}$$

and is in the direction from the north pole to the south pole (*i.e.*, opposite to the direction of the lines of force through the magnet).

(c) Prove that when $P$ is outside the magnet but very close to its north pole, and the magnet is so long that $\theta'$ is very small, the resultant magnetizing force at $P$ is

$$H'_o = 2\pi\sigma \tag{11b}$$

(d) Prove that when two such cylindrical magnets are placed end to end, with the north pole of one separated from the south pole of the other by a narrow air-gap, the resultant magnetizing force in the air-gap is

$$H_r = 4\pi\sigma \tag{11c}$$

(e) Prove that the mechanical force of attraction between these adjacent north and south poles is

$$f = 2\pi\sigma^2 S = \frac{2\pi m^2}{S} \tag{12}$$

where $m$ is the total strength of each pole and $S$ is the cross-section of the magnet.

(f) Show that this force of attraction is also

$$f = \frac{B^2 S}{8\pi} \tag{12a}$$

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where $B$ is the resultant flux density in the air-gap, which in turn is equal to the resultant magnetizing force $H_r$ in this gap.

(g) Why is the force exerted by the pole of one magnet on the pole of the other not equal to the resultant magnetizing force $H_r$ multiplied by the strength $m$ of this pole?

**135. Practical Calculation of the Mechanical Forces Exerted on Magnetic Bodies by a Magnetic Field.**—A more practical method than that outlined in the preceding article, of expressing the mechanical force exerted on a magnetic body in a magnetic field, is to express the work done by this force, in producing an infinitesimal displacement of this body, in terms of the change in the reluctance of the magnetic field in which it is located, due to the given displacement.

Imagine a magnetic body of any size or shape in a magnetic field, and let $\mathcal{R}$ be the total reluctance of the field, and let $\varphi$ be the total magnetic flux. On the assumption that the reluctance of the field is independent of the value of flux (*i.e.*, constant permeability) the total magnetic energy of the field is, from equation (10) of Article 116,

$$W = \frac{1}{8\pi}\,\mathcal{R}\varphi^2$$

Let $d\mathcal{R}$ be the increase in the reluctance $\mathcal{R}$ corresponding to a displacement $dx$ of the given magnetic body in any chosen direction. Then the *increase* energy in the magnetic field as the result of this displacement, *i.e.*, the increase in $W$ corresponding to the displacement $dx$, is

$$dW = \frac{\varphi^2}{8\pi}\,d\mathcal{R}$$

The increase $dW$ in the energy of the magnetic field, since it is due entirely to the motion of the given magnetic body, must be equal to the work done by the force which moves this body *against* the force exerted on the body by the magnetic field. Let $f_s$ be the component, in the direction of $dx$, of the force exerted *by* the field *on* the given body, then

$$dW = -f_s dx$$

(The minus sign here signifies that when the body moves in the direction of the force exerted by the field, the field does work and therefore loses energy, *i.e.*, the increase in $W$ is negative.) Equating these two expressions for $dW$ gives

$$f_s = -\frac{\varphi^2}{8\pi}\,\frac{d\mathcal{R}}{dx} \qquad (13)$$

The minus sign in this expression shows that the direction of the force exerted by a magnetic field on a magnetic body is always in the direction in which the body must move in order to *decrease* the total reluctance of the field. That is, a magnetic body in a magnetic field always tends to move into such a position as will result in a *decrease* in the reluctance of the field.

Equation (13) is deduced on the specific assumption that the reluctance $\mathfrak{R}$, for a given relative position of the various parts of the magnetic circuit, is independent of the value of the flux $\varphi$ in this circuit. When the magnetic circuit is wholly of iron or other ferromagnetic substance this condition does not hold. However, when the major portion of the reluctance of the circuit is in air or other non-magnetic substance, equation (13) may be applied even though the greater part of the volume of the magnetic circuit is in iron.

A simple application of equation (13) is to the calculation of the force exerted on the plunger of an iron-clad electromagnet, such as shown in Fig. 78. Let $x$ be the distance between the stop and the upper end of the plunger, and let $S$ be the cross-section of the air-gap. On the assumption that the lines of force in the air-gap are uniformly distributed and parallel to the axis of the plunger, the reluctance of the gap is then $\frac{x}{S}$. When the plunger is displaced downward a distance $dx$ this reluctance therefore increases by an amount $\frac{dx}{S}$.

That portion of the length of the plunger which carries the flux also decreases by an amount $dx$, but since the permeability of the plunger is 1000 times or more that of air, the corresponding decrease in reluctance is negligible in comparison with the increase in the reluctance of the air-gap. Hence the increase in the total reluctance of the magnetic circuit is substantially

$$d\mathfrak{R} = \frac{dx}{S}$$

This substituted in equation (13) gives for the total pull *upward* on the plunger (opposite to $dx$) the value

$$f = \frac{\varphi^2}{8\pi S} \qquad \text{dynes} \qquad (14)$$

Or, since $\varphi = BS$, where $B$ is the flux density in the gap (the lines of force being assumed uniformly distributed and parallel

to the axis of the plunger), the force acting upward on the plunger may also be written

$$f = \frac{B^2 S}{8\pi} \qquad \text{dynes} \qquad (14a)$$

For practical calculations it is usually convenient to express $B$ in kilolines per square inch, and $S$ in square inches. When these units are employed this formula becomes

$$f = 0.0139 B^2 S \qquad \text{pounds} \qquad (14b)$$

Compare with the formula deduced in Problem 4 of Article 116.

In order to apply equations (14) to the calculation of the pull on the plunger or armature of an electromagnet of a given number of ampere-turns, it is of course necessary to determine first the value of the flux $\phi$ in terms of these ampere-turns. When the air gap is short, and the leakage therefore small, this may readily be done by the procedure outlined in Article 91. However, when the air gap is large, as is usually the case, the calculation of the flux produced by a given number of ampere-turns, or the ampere-turns required to produce a given flux, is not so simple. See the articles on *Electromagnet Windings* and *Electromagnets* in Pender's *Handbook for Electrical Engineers*.

Problem 7.—The magnetic circuit of the electromagnet shown in Fig. 78 is made of cast steel. The plunger and stop are each 1 inch in diameter. The mean length of the lines of force in the several parts of the magnetic circuit when the upper end of the plunger is in contact with the stop are as follows:

> In plunger, 7 inches.
> In stop, 1 inch.
> In yoke, 15 inches.

The mean cross-section of the yoke is 6 inches. The air-gap between the yoke and plunger where the latter passes through the lower part of the yoke has a length of $\frac{1}{32}$ inch and a cross-section of 5 inches. The winding has 400 turns. Assume no leakage. See Fig. 76 for $B$-$H$ curve.

(a) What current is requ'red in the w'nding in order to hold the plunger up against the stop with a force of 100 pounds? (b) What current is required in the winding in order to produce this same force on the plunger when the upper end of the latter is $\frac{1}{4}$ inch from the stop? (Assume the lines of force in this air-gap to be straight, parallel and uniformly distributed.) (c) Were the plunger and stop each 2 inches in diameter instead of 1 inch (the clearance at the hole in the yoke being kept the same as before, namely $\frac{1}{32}$ inch), what would be the current required to produce a force of 100 pounds on the plunger when the upper end of the latter is in contact

with the stop? (*d*) When the upper end of the plunger is ¼ inch from the stop? (*e*) For the same current in the winding, will the pull increase or decrease with the diameter of the plunger? (*f*) Explain why your answer is not inconsistent with the fact that for the same flux in the plunger the pull is *greater* the *smaller* the diameter of the plunger.

*Answer.*—(*a*) 2 amperes. (*b*) 21 amperes. (*c*) 0.9 ampere. (*d*) 10 amperes. (*e*) The pull increases with the diameter of the plunger.

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